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Two sided long-time optimization singular control problems for Itô-Diffusions and Lévy processes with applications to mean field games

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RESUMEN

En esta tesis, nos enfocamos en el estudio de problemas de control singular de horizonte infinito y de variación acotada para procesos de Lévy y difusiones de Itô con aplicaciones a juegos de campo medio. Nos interesa caracterizar y, en algunos casos, proporcionar una representación explícita de las estrategias óptimas y dar condiciones suficientes para la existencia y unicidad de estrategias de equilibrio. Debido a la naturaleza diferente de los procesos de interés, podemos dividir en dos ramas nuestros métodos:

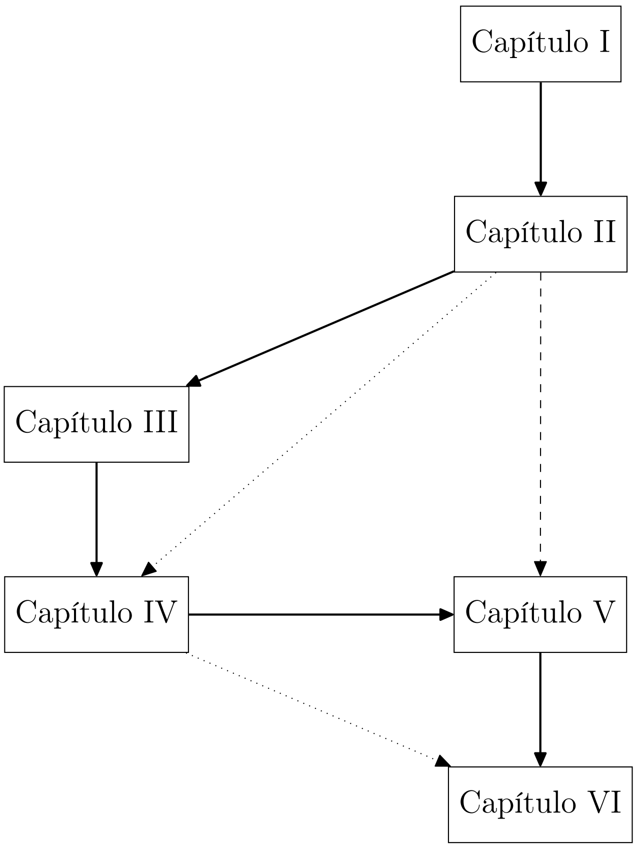
- Para difusiones de Itô, se plantea un problema de control singular ergódico de variación acotada. Se da un teorema de verificación para las estrategias óptimas y se encuentra un candidato como solución de una ecuación diferencial con condiciones de frontera. Luego, la estrategia de equilibrio del juego de campo medio (MFG) se caracteriza como la raíz de una ecuación, y se proporcionan condiciones fáciles de verificar para probar su existencia y unicidad.
- Para procesos de Lévy, trabajamos tanto con un problema de control singular descontado como con un problema ergódico de variación acotada. Como antes, se plantea un teorema de verificación. Como en este caso, la ecuación de Hamilton-Jacobi-Bellman (HJB) es una ecuación integro-diferencial, el curso de acción es probar la relación entre estos problemas y los juegos de Dynkin (lo cual es conocido para otras familias de procesos). Luego, utilizamos esta relación para probar la existencia de una estrategia óptima descontada para el problema descontado y usamos el límite abeliano para caracterizar la estrategia óptima ergódica. Finalmente, se plantea un teorema de punto fijo para el MFG para demostrar la existencia de un equilibrio.

La tesis se desarrolla de la siguiente manera. En los capítulos 1 y 2, se presenta la introducción de los problemas (omitiendo la mayor parte del análisis histórico, ya que preferimos dar uno detallado en cada capítulo), el marco teórico y la mayoría de las notaciones. En el Capítulo 3 tratamos un problema de control singular ergódico bilateral para difusiones de Itô y en el Capítulo 4 añadimos un componente MFG al problema. En el Capítulo 5 tratamos tanto un problema de control singular ergódico como descontado, bilateral para procesos de Lévy, y en el Capítulo 6 añadimos un componente MFG al problema.
Las contribuciones de esta tesis se pueden resumir de la siguiente manera:

- En el Capítulo 3 extendemos los resultados de [Alvarez(2018)] a un conjunto de controles más general, y en el Capítulo 4 analizamos el problema asociado al MFG. Los resultados fueron publicados en [Christensen et al. (2023)].

- En el Capítulo 5 demostramos que para procesos de Lévy el problema de control singular descontado bilateral está asociado a un juego de Dynkin. Además, probamos que el límite abeliano es válido y damos ejemplos explícitos. Los resultados se pueden encontrar en el preprint [Mordecki and Oliú (2024)].
- En el Capítulo 6 añadimos un componente MFG al problema planteado en el Capítulo 5, y utilizando algunas propiedades de regularidad en el juego de Dynkin, demostramos que existe un equilibrio para el problema descontado y, usando el límite abeliano, demostramos que también existe un equilibrio para el problema ergódico del MFG.

Para los lectores familiarizados con el tema, presentamos dos formas alternativas de leer esta tesis.



Como la literatura es más escasa, el lector puede saltar del Capítulo 2 al Capítulo 5 donde comienza el estudio de los procesos de Lévy.
Por otro lado, si el lector está bien versado en la relación entre juegos de Dynkin y problemas de control singular bilateral y desea leer aplicaciones más prácticas, los resultados del Capítulo 5 no serán difíciles de creer y el lector puede saltar del Capítulo 2 al 4 y luego al Capítulo 6

donde se tratan los problemas MFG.

Palabras claves:
Procesos de Lévy, Difusiones de Itô, Control singular ergodico, Control singular descontado, Juegos de Campo Medio, Equilibrios de Nash, Juegos de Dynkin.

ABSTRACT

In this thesis, we focus on the study of infinite horizon singular control problems of bounded variation for Lévy processes and Itô-diffusions with applications to mean field games. We are interested in characterizing and in some cases, giving an explicit representation of the optimal strategies and provide sufficient conditions for the existence and uniqueness of equilibrium strategies. Due to the different nature of the processes of interest we can divide in two branches our methods:

- For Itô-diffusions, an ergodic singular control problem of bounded variation is posed. A verification theorem for the optimal strategies is given and a candidate is found as the solution of a differential equation with boundary conditions. Then, the mean field game (MFG) equilibrium strategy is characterized as the root of an equation and easy to check conditions are given in order to prove its existence and uniqueness.
- For Lévy processes, we work both with a discounted and an ergodic singular control problem of bounded variation. As before, a verification theorem is posed. As in this case, the Hamilton-Jacobi-Bellman (HJB) equation is an integro differential equation, the course of action is to prove the relationship between these problems and Dynkin games (which is known for other families of processes). Then we use this relationship to prove the existence of a discounted optimal strategy for the discounted problem and use the abelian limit to characterize the ergodic optimal strategy. Finally a fixed point theorem is posed for the MFG to prove the existence of an equilibrium.

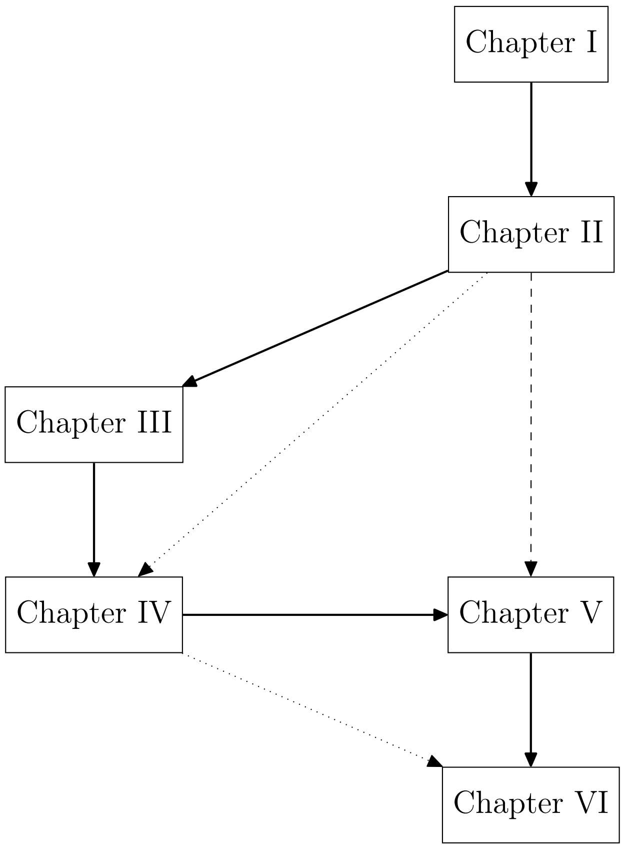
The thesis is developed as follows. In chapters 1 and 2, the introduction of the problems (omitting most of the historical analysis as we prefer to give a detailed one in each chapter), the theoretical framework and most of the notations are given. In Chapter 3 we treat a two sided ergodic singular control problem for Itô-diffusions and in Chapter 4 we add a MFG component to the problem. In Chapter 5 we treat both a two sided ergodic and discounted, singular control problem for Lévy processes and in Chapter 6 we add a MFG component to the problem.
The contributions of this thesis can be summarized as follows:

- In Chapter 3 we extend the results of [Alvarez(2018)] to a more general set of controls and in Chapter 4 we analyze the MFG associated problem. The results were published in [Christensen et al. (2023)]
- In Chapter 5 we prove that for Lévy processes the two-sided discounted singular control problem is associated to a Dynkin game. Furthermore we prove that the abelian

limit holds and give explicit examples. The results can be found in the preprint [Mordecki and Oliú (2024)]

- In Chapter 6 we add a MFG component to the problem posed in Chapter 5, using some regularity properties in the Dynkin game we prove that there is a equilibrium for the discounted problem and using the abelian limit we prove that there is also an equilibrium for the ergodic MFG problem.

For the readers who are familiar with the topic we present two alternative ways of reading this thesis.



As the literature is more scarce, the reader can jump from Chapter 2 to Chapter 5 where the study of Lévy processes begin.
On the other hand, if the reader is well versed in the relationship between Dynkin games and two-sided singular control problems and want to read more practical applications, the results of Chapter 5 will not be hard to believe and the reader can jump from Chapter 2 to 4 and then to Chapter 6 where the MFG problems are treated.

Keywords:
Lévy processes, Itô-diffusions, Ergodic singular control, Discounted singular control, Mean

field games, Nash equilibrium, Dynkin games.

Contents

1 Introduction **1**
1.1 Two-sided stochastic singular control problems . . . 1
1.1.1 The ergodic problem . . . 2
1.1.2 The ϵ-discounted problem . . . 3
1.1.3 Reflecting controls . . . 3
1.2 Mean field games . . . 4
1.2.1 A finite time horizon formulation of a mean field game . . . 4
1.2.2 An infinite horizon stationary formulation of a mean field game . . . 5
1.2.3 N-player symmetric problem . . . 6
1.3 Contributions of the thesis . . . 7
1.4 Thesis structure . . . 7

2 Theoretical framework and formulation of the main problems **11**
2.1 General framework . . . 11
2.1.1 Filtered probability space . . . 11
2.1.2 Martingales and local martingales . . . 13
2.2 Underlying processes . . . 14
2.2.1 Introduction . . . 14
2.2.2 Stochastic integration with respect to the Brownian motion . . . 15
2.2.3 Itô-diffusion . . . 16
2.2.4 Lévy processes . . . 18
2.3 Stochastic calculus . . . 21
2.3.1 Semimartingales and stochastic integration . . . 21
2.3.2 Stochastic integration . . . 22
2.3.3 Itô formula . . . 23
2.4 Excessive functions and the infinitesimal generator . . . 24
2.5 Reflection on intervals . . . 26

2.5.1 Deterministic case 27
2.5.2 Stochastic general framework 28
2.5.3 Itô-diffusion 28
2.5.4 Lévy processes 29
2.6 Regenerative theory 29
2.7 Stochastic singular control problems 30
2.7.1 The ergodic problem 31
2.7.2 The ϵ-discounted problem 31
2.8 Relationship with Dynkin games 32
2.9 Mean field games for two-sided singular control problems 33
2.9.1 Itô-difusions 33
2.9.2 Lévy processes 34

3 Two-sided ergodic control for Itô-diffusions 36
3.1 Introduction 36
3.2 Framework for the Ergodic Control problem 38
3.2.1 Itô-diffusion 38
3.2.2 Cost function 40
3.3 Optimality within reflecting controls 41
3.3.1 Conditions for optimal barriers 41
3.3.2 Adjoint free boundary problem, existence and uniqueness of optimal reflecting controls 45
3.4 Optimality within admissible controls 47
3.5 Examples 50
3.5.1 Orstein–Uhlenbeck process 50
3.5.2 Brownian motion with drift 52

4 Two-sided ergodic singular mean field games for Itô-diffusions 54
4.1 Introduction 55
4.2 Setting 60
4.3 Characterization of the MFG equilibrium 61
4.3.1 Conditions for optimality and equilibrium 61
4.3.2 The multiplicative case 64
4.4 Examples 67
4.4.1 Examples with multiplicative cost 67
4.4.2 "Follow the market" examples 69

4.5 Approximation of Nash equilibria in symmetric N-player games with mean field interaction . . . 73

5 Ergodic and discounted singular control problems for Lévy processes 79

5.1 Introduction . . . 80
5.2 Setting . . . 88
5.2.1 Lévy processes, controls and cost functions . . . 88
5.2.2 The ergodic and discounted control problems . . . 91
5.2.3 Main results . . . 91
5.3 Preliminary results . . . 92
5.4 The associated Dynkin game . . . 99
5.4.1 Introduction and properties . . . 99
5.4.2 The value function . . . 103
5.5 Construction of the candidate . . . 108
5.5.1 Properties of the primitive of the value function . . . 108
5.5.2 Continuity properties of reflecting controls . . . 109
5.5.3 Proofs of Theorems 5.2.1 and 5.2.2 . . . 113
5.6 Examples . . . 117
5.6.1 Introduction . . . 117
5.6.2 Ergodic problem with absolute value cost for Poisson Compound Process with two-sided exponential jumps . . . 118
5.6.3 Discounted problem with quadratic cost for a Compound Poisson process with two-sided exponential jumps and Gaussian Noise . . . 121
5.6.4 Ergodic problem with quadratic cost for strictly stable processes . . . 122

6 Stationary mean field games for two-sided Control Lévy problems 125

6.1 Introduction . . . 126
6.2 Framework and main results . . . 128
6.2.1 Setting . . . 128
6.2.2 Main results . . . 130
6.3 Mean field equilibrium of the ϵ-discounted problem . . . 131
6.3.1 The adjoint Dynkin game . . . 132
6.3.2 Fixed point . . . 134
6.4 Mean field equilibrium for the ergodic problem . . . 137
6.4.1 Ergodic results . . . 137
6.4.2 Proof of Theorem 6.2.4 . . . 140

6.5 Examples 141
6.5.1 MFG Discounted problem with quadratic cost for a Compound Poisson process with two-sided exponential jumps and Gaussian Noise 142
6.5.2 Ergodic MFG for Strictly stable process 142
6.6 Approximation of Nash equilibria in symmetric N-player games with mean field interaction 144
A.1 Reflection of Itô diffusions on intervals 147
A.2 Regenerative properties of Lévy processes 151
A.3 The problem of the first exit time of an interval for some Lévy processes 154
A.3.1 Two-barrier problem for Poisson Compound Process with two-sided exponential jumps and negative mean 155
A.3.2 Discounted two-barrier problem for a Compound Poisson process with two sided exponential jumps, Gaussian Noise and non-zero mean 156

Bibliography **162**

Chapter 1

Introduction

In this thesis, we study *two-sided stochastic long time singular control optimization problems* and *stationary mean field games* (MFGs for short) for Itô-diffusions and Lévy processes. The aim of the following introduction is to describe the problems to be treated and the structure of the chapters to give a general panorama of the thesis. In fact, we have chosen to relegate most of the analysis of the selected literature to each chapter in order to make this manuscript more readable. Moreover, the reader who is not familiar with the mathematical theory mentioned in the introduction should be able to understand the thesis after reading Chapter 2.

1.1 Two-sided stochastic singular control problems

The control problems of our study are more commonly known as *bounded variation, stochastic singular control optimization problems* but, as we are working in the real line, we can make a distinction between the cost of the increasing part of the control and the cost of the decreasing one. In stochastic bounded variation control, the displacement of the state caused by the control is of bounded variation. Moreover, we say it is singular when the optimal control is the reflection inside an interval. The reason for this name is that for the Brownian motion (not necessarily for every Lévy processes) the trajectory of the reflected controls in intervals are singular with respect to the Lebesgue measure.
To keep it short:

- There is a filtered probability space, in other words, information is revealed over time.
- An underlying uncontrolled stochastic process which in this thesis is an Itô-diffusion or Lévy process.
- There is a set of *admissible control processes* which is a family of increasing pairs (U, D) of adapted processes which control the process (the process U push upwards and D

downwards). Notice that we could have defined *admissible controls* as a bounded variation process. However, we want to easily distinguish the increasing and decreasing parts in our control problems.

- A function which depends on the starting point x of the process and the controls chosen.
- The particularities of our problems make the optimal control a reflecting strategy in an interval, that is, there is a couple of points so that the best control is to make *the smallest push* to keep the process in the interval defined by the points.

As applications of singular controls problems we mention, studies focusing on cash flow management, recapitalization or a combination of both, while considering risk neutrality. Moreover, in the ergodic case, singular control problems also have applications in sustainable harvesting policies. To avoid redundancy, we refer to 3.1 and 5.1 for a selected analysis of the literature.

1.1.1 The ergodic problem

In the two-sided ergodic control problem, we have an integral cost averaged over infinite time, thus rendering unimportant any initial finite action. To be more specific, the problem consists of finding a pair (U^*, D^*) that minimizes the expression

$$\limsup_{T\to\infty} \frac{1}{T}\mathbf{E}_x\left(\int_0^T c(X_s^\eta)ds + q_u U_T + q_d D_T\right),$$

where $\mathbf{E}_x$ is the mean when the uncontrolled process starts at x , $X^\eta = \{X_t^\eta\}_{t\geq 0}$ is the controlled process. The process $\eta = (U, D)$ is an *admissible control*, c is a running cost function, q_u and q_d are positive constants representing the cost of exercising the controls. Two roadmaps are taken in this thesis:

- When the underlying process is an Itô-diffusion, we give a verification theorem in the form of a Hamilton-Jacobi-Bellman (HJB for short) equation that characterizes the optimal control. We find it explicitly through analytic means. Although original for our specific problem this is not an unusual technique (see, mostly [Alvarez(2018)]).
- When there is an underlying Lévy process, we give a verification theorem in the same way as the previous case. However as the HJB equation in this case is an integro-differential equation, our technique is to study a *discounted problem*, prove that the abelian limit holds and the limit characterizes our solution. This allows us to choose a more tractable set of controls where we use the results of [Andersen et al. (2015)] to obtain the optimal controls.

1.1.2 The ϵ-discounted problem

From an operational point of view, the discount rate added to a control problem allows us to work in infinite horizon. The concept has origins applied to physics (see [Peskir and Shiryaev (2006), Chapter V, page 122]). From a financial point of view, the exponential discount rate occurs when a financial agent discounts future costs by the same factor. The ϵ-discounted problem we are interested in this thesis is to find an *admissible controls* (U^*, D^*) that minimizes the expression:

$$\mathbf{E}_x \left(\int_0^\infty e^{-\epsilon s} \left(c(X_s^{U,D}) ds + q_u dU_s + q_d dD_s \right) + q_u u_0 + q_d d_0 \right).$$

The literature for this problem when the underlying process is an Itô-diffusion is vast (as it is exposed in 3.1). For this reason, we are only interested in this problem when there is an underlying Lévy process. Our roadmap is to

- Postulate a verification theorem in the form of a HJB equation
- Prove that the problem has an associated Dynkin game. This relationship is know under different frameworks, this is exposed in Chapter 5.
- Find a candidate of the optimal control with the equlibrium points of the Dynkin game.

1.1.3 Reflecting controls

As mentioned earlier, in the control problems we work in this thesis, the best course of action is to choose an adequate couple of barriers $a \le b$ (the inequality is strict if the process has unbounded variation) and with *the minimal possible push*, preventing the process from leaving $[a, b]$. As it will be explained in 2.5, these controls are well-defined in the sense that they exists and are unique. Without entering into technical details, for Lévy processes, the reflecting controls and the reflected process can be intuitively constructed from the one-sided reflections in the following way:

- Assume $a = 0,\ b > 0$ and the starting point of the underlying process $X = \{X_t\}_{t \ge 0}$ is $x \in [a, b)$.
- Define

$$\tau_0 = \inf\{t : X_t^0 \ge b\}, \quad \text{where } X^0 \ \text{ is the reflected process on } [0, \infty),$$

  and the process $U = \{U_t\}_{t \ge 0}$ as the one sided reflecting control of X in the half line $[0, \infty)$.

- Once the process reaches b, take X_t^b as the reflection of the process $X_t - X_{\tau_0} + b$ in $(-\infty, b]$ and define

$$\tau_1 = \inf\{t \geq \tau_0 : X_t^b \leq 0\},\ D_{\tau_0} := X_{\tau_0} - b,\ D_t = D_{\tau_0} + D_t^b,$$

  where D_t^b is the one sided reflection of the process in $(-\infty, b]$.
- Once the process reaches τ_1 switch again to the reflection in the lower barrier.

Their regenerative properties allow us to work in the ergodic case (see 2.6). For Lévy processes the construction is essentially deterministic (see 2.5.4). The case of Itô-diffusions and more general stochastic processes is explained in 2.5.2 and 2.5.3.

1.2 Mean field games

Paraphrasing [Maschler et. al. (2013)], game theory is the name given to the methodology of using mathematical tools to model, analyze, predict and even suggest situations of interactive decision making. These are situations involving several decision makers (called players) with different goals, in which the decision of each one affects the outcome for all the decision makers. In game theory, a *strategic game* or N-player game, consists of a set of players, a strategy set for each player, and an outcome corresponding to each vector of strategies. One of the theoretical objectives of strategic games is the search of *Nash equilibrium*. That is, for an N-player game, a N-nuple of strategies such that no player has a profitable unilateral deviation from it.
In the search of a Nash equilibrium, when a large quantity of players try to minimize their respective cost functions the simplification that is natural is to treat the *asymptotic problem*. This new problem where the players are replaced by a measure or in some cases a function is called *mean field game* or for short MFG. A commentary on selected mean field games problem, mostly similar to the ones of our interest, is in chapters 4 and 6. As a reference, with many relevant historical examples, we cite two-volume monograph by [Carmona and Delarue (2018)]. For one of the first works on the matter, see [Lasry and Lions (2006)]. As applications, for instance, in finance and energy systems see [Carmona(2021)], in traffic management and social dynamics see [Festa and Göttlich (2018)], in machine learning see [Subramanian & Mahajan (2019)].

1.2.1 A finite time horizon formulation of a mean field game

For the reader who is not familiar with this theory, we recommend not reading this subsection until finishing reading Chapter 2. This material is extracted from [Carmona and Delarue (2018), page 132]. Let us consider

- A complete filtered probability space with an associated d-dimensional Brownian motion $W = \{W_t\}_{t\geq 0}$.
- A starting initial condition ζ measurable with respect to the initial σ-algebra and that has second moments.
- A set $\mathcal{A}$ of adapted processes, called *set of admissible strategies*, with *reasonable* measurability and integrability conditions taking values in a convex set $A \subset \mathbb{R}^d$.
- A continuous function $c : [0, T] \times \mathbf{R}^d \times \mathcal{P}(\mathbb{R}^d) \times A \to \mathbb{R}$ called *running cost* and a continuous function $g : \mathbf{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ called *terminal cost.*

The MFG problem can be summarized as follows

(i) For each fixed deterministic flow $\mu = \{\mu_t\}_{0\leq t\leq T}$ on $\mathbb{R}^d$, find $\alpha^* \in \mathcal{A}$ that minimizes the expression

$$\mathbf{E}\left(\int_0^T c(t, X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right),$$

where $X^\alpha = \{X_t^\alpha\}_{0\leq t\leq T}$ is the controlled process.

(ii) Find a flow μ, such that $\text{Law}(X_t^\mu) = \mu_t$, for all $t \in [0, T]$ if the controlled process X^μ is a solution of the problem above. This μ is called MFG *equilibrium.*

Here, the process X can be thought as the private state of a representative player. The methods to solve these problems are diverse. One common technique to solve these problems is to enlarge the space and use *forward backwards stochastic differential equation*, FBSDE for short, to characterize the optimal control, commonly and respectfully called *master equation* (see [Carmona and Delarue (2018), 5.7.2]). It is clear that variations of the problems exists, such as considering discountinuous processes or changing the way that control interacts with the cost (see Chapter 4 for a wider analysis).

1.2.2 An infinite horizon stationary formulation of a mean field game

The previous example can be extended to the case $T \to \infty$. However we proceed to restrict our attention to the mean field games (MFGs) that are of our interest in this thesis. As the previous case there is an initial optimization problem. These problems are the same (except some technical restrictions in the set of admissible controls) as the ones postulated in 1.1 with the exception that now c depends on a second parameter y representing the stationary state of the infinite players. This simplifies the optimization problem greatly and the stationary assumption is justified for example in 4.5. Secondly, we assume that in this case the parameter is of the form:

$$y = \int_\Omega h(x) d\mu(x),$$

for the stationary measure μ. To solve the second part of the MFG, that is to find a MFG *equilibrium* we take two routes depending on the underlying process:

- When the process is an Itô-diffusion, we observe that the adjoint HJB that defines the optimal control now depends on a parameter. This can be translated into a non-linear equation.
- When the process is Lévy, a fixed point theorem is used for the ϵ-discounted case an the abelian limit for the ergodic one.

1.2.3 N-player symmetric problem

In contexts similar to ours, the state of the system is the aggregation of private states of individual players modelled by N-stochastic processes. There is a family of adapted stochastic processes $\mathcal{A}$, taking values in a set A (usually a metric space) called *admissible controls*. The set A, can be dependent on the player but due to the interests of this thesis we need certain symmetry, thus we assume the set of possible actions is independent of the player. If each player i chooses an admissible control α^i, that person will pay a cost depending on $\alpha^1, \dots, \alpha^N$ of the form:

$$J^i(\alpha^i, \alpha^1, \dots, \alpha^{i-1}, \alpha^{i+1}, \dots, \alpha^N).$$

or $J^i(\alpha^i, \alpha^{-i})$ for short. In our ergodic N-player problem, the controls are of the form $\eta^i = (U^i, D^i)$ and the cost for each player $i = 1, \dots N$, is of the form:

$$J^i(\eta^i, \eta^{-i}) = \limsup_{T\to\infty} \frac{1}{T} \mathbf{E}_x \left(\int_0^T c(X_s^{\eta^i}, \eta^1, \dots, \eta^N) ds + q_u U_T^i + q_d D_T^i \right).$$

For the ϵ-discounted N-player control problem, it is clear how J^i will look. As said before, in this setting, in the search of an *approximate* Nash equilibrium the need to work with MFGs arises. To make the objectives clear, but without going into technical details we give the following definitions.

Definition 1.2.1. *A vector of admissible controls $\alpha = (\alpha^1, \dots, \alpha^N)$ is called*

(i) *A* Nash equilibrium *if*

$$J^i(\alpha^i, \alpha^{-i}) \leq J^i(\hat{\alpha}^i, \alpha^{-i}), \ \forall \hat{\alpha}^i \in \mathcal{A}, \ \forall i$$

(ii) *An* ϵ-Nash equilibrium *if*

$$J^i(\alpha^i, \alpha^{-i}) \leq \epsilon + J^i(\hat{\alpha}^i, \alpha^{-i}), \ \forall \hat{\alpha}^i \in \mathcal{A}, \ \forall i$$

(iii) *An* open loop equilibrium *if it is a Nash equilibrium under the additional assumption that* $\alpha^{-i} = (\alpha^1, \dots, \alpha^{i-1}, \alpha^{i+1}, \dots, \alpha^N)$ *does not change trajectories once the player* i *changes trajectories.*

Basically, definition (i) is what one searches in a N-player control game. Definition (ii) is what ones aspires to get with a MFG equilibrium, that is ϵ as function of the number of players that tends to zero when $N \to \infty$. Finally, although we are not interested in this thesis but it will be mentioned in some references, the definition (iii), which is more mathematically tractable, could be thought of as the abstraction where the other players ignore the change of action of an individual players.

1.3 Contributions of the thesis

The contributions of this thesis can be summarized as follows

- In Chapter 3 we generalize a two-sided ergodic singular control problem proposed in [Alvarez(2018)] when the underlying process is an Itô-diffusion. Moreover, in Chapter 4 we add a MFG component, characterize the equlibrium points and give conditions for existence and uniqueness. The results are published in [Christensen et al. (2023)].
- Chapter 5, is the longest one as we treat a problem with scarce literature. That is two-sided ergodic and discounted singular control problems when there is an underlying Lévy process. We use an associated Dynkin game to characterize the optimal controls for the discounted problem and prove that the abelian limit holds. The results are in the preprint [Mordecki and Oliú (2024)].
- In Chapter 6 we add a MFG component to the problem in Chapter 5, the main tool is to use a fixed point theorem in the Dynkin game and the abelian limit for the ergodic case. The results here have not yet been published.

1.4 Thesis structure

Finally, we briefly describe the structure of the thesis.

In Chapter 2 we write down most of the notations and the minimal necessary theory needed for an understanding of most of the thesis, except for some of the results of the Appendix.

- In Section 2.1 we give some basic definitions with the aim of formalizing the concept of *information up to a fixed time.*
- In Section 2.2 we define and give some useful properties of the processes that will be studied in this thesis.
- In Section 2.3 we bring a brief summary of the tools of stochastic calculus that will be used.
- The topic we treat in Section 2.4 is *excessive functions*, that is, the probabilistic view of harmonic theory. Obviously we just give a brief resume of this rich topic as it is crucial in optimization problems.
- In Section 2.5 we explain the general characteristics of the optimal strategies of the problems in our thesis, that is, *reflecting strategies.*
- In Section 2.6 we enumerate the results of regenerative theory that will be used in this thesis. As our problems are of infinite horizon, it is natural to use this theory.
- In Section 2.7, we formalize most of the setting of singular control problems.
- The relationship of Dynkin games and two-sided singular control problems, which was mentioned earlier, is better summarized in Section 2.8. This is one of the central topics of Chapter 5 and here we briefly explain the Dynkin game that will be used.
- Finally in Section 2.7, we give the notations and postulate the MFGs of our interest.

In Chapter 3 we treat an ergodic two-sided singular control problem when the problem is an Itô-diffusion. The objective is to generalize [Alvarez(2018)] to a more general set of controls.

- In the introduction 3.1 we give a brief summary of some selected works.
- In Section 3.2 we laid down the notations and problem of this Chapter.
- In Section 3.3 we write the most important results of [Alvarez(2018)], that is, the characterization, existence and uniqueness of the optimal controls within the family of reflecting controls (Theorem 3.3.3).
- Moreover in 3.4 we extend the results of the previous section to a more general family of controls (Theorem 3.4.2).
- Finally, in Section 3.5 we give some examples when the underlying processes are Orstein-Ulhenbeck and Brownian motion with drift.

In Chapter 4 we add a MFG component to the problem posed in the previous chapter.

- In Section 4.1, we mention some of the applications of MFGs and give a brief resume of some selected works.

- In Section 4.2 we are concerned with writting down the setting.
- In Section 4.3 we characterize the MFG equilibrium in Theorem 4.3.3. Furthermore we give conditions for the existence and uniqueness of MFG equilibrium points for a particular class of running costs (Proposition 4.3.5).
- In Section 4.4 we give explicit examples when the underlying processes are Orstein-Ulhenbeck and Brownian motion with drift.
- Finally, in Section 4.5 we study the approximation for the N-player control problem. The main result here is Theorem 4.5.1.

In Chapter 5 we study an ergodic and a discounted two-sided, long time average singular control problem when the underlying process is a Lévy process.

- As usual, in Section 5.1 we provide a selected analysis of the literature. In this case we are concerned with the relationship between the problems of the chapter and Dynkin games.
- In Section 5.2 we give the notations and main results of the chapter.
- In Section 5.3 we give verification results that provides sufficient conditions for controls to be optimal for both the ergodic and the discounted problems. See theorems 5.3.6, 5.3.5, 5.3.4 and 5.3.3.
- In Section 5.4, we study the properties of an associate Dynkin game, mostly harmonic properties in Proposition 5.4.1 and in propositions 5.4.4, 5.4.5 and 5.4.6 regularity properties.
- In Section 5.5, in Theorem 5.2.1, we show that the solution of the discounted problem is the solution of a Dynkin game in the sense that the continuation region of the Dynkin game is an interval whose extremes define a two-sided reflecting optimal control for the discounted problem. For that endeavor, several properties referencing the reflecting controls are studied. Furthermore, we prove that the abelian limit holds (see Theorem 5.2.2).
- In Section 5.6, we present three examples, the first two when the driving Lévy process are Compound Poisson process with two-sided exponential jumps with and without Gaussian component, the third involving a strictly stable process with finite mean.

In Chapter 6 we incorporate a mean field game dependence into the two-sided discounted and ergodic problems posed in the previous chapter.

- In Section 6.1 we give a a brief resume of some selected articles.
- In Section 6.2 we define the framework and the main results of this chapter, that is, we characterize and give conditions for the MFG equilibrium to hold in Theorem 6.2.3 and Theorem 6.2.4.

- In Section 6.3 we use the adjoint Dynkin game to prove that there is a MFG equilibrium for the ϵ-discounted control problem, for that endeavor we use Brouwer Fixed Point Theorem to prove Theorem 6.2.3. That is proving that the MFG ϵ-discounted MFG equilibrium point exists and it is the solution of an HJB equation.
- In Section 6.4 we use regenerative theory to prove Theorem 6.2.4. To keep it short, we prove that equilibrium points in the discounted case have a convergent subsequence to a MFG equilibrium for the ergodic problem.
- In Section 6.5 we provide numerical examples.
- Finally in Section 6.6 we study the convergence of the N-player game to the MFG for both problems (see Theorem 6.6.1).

Chapter 2

Theoretical framework and formulation of the main problems

Abstract

In this chapter, we write the minimal necessary theory needed for a complete understanding of this thesis (with the exception of the Appendix). We also explain the problems to be studied in the rest of the thesis.

The objectives of this chapter, assuming that the reader has basic knowledge of measure theory and probability, are:

- expose the minimum necessary theory used in the thesis,
- provide most of the definitions that are used and
- explain the problems to be studied in subsequent chapters.

2.1 General framework

2.1.1 Filtered probability space

In this thesis we study optimization problems where the actor must choose the adequate times to act. Thus, in this subsection, we formalize the term *information up to* time $t \geq 0$. As usual, we use the triad $(\Omega, \mathcal{F}, \mathbf{P})$ to denote a probability space.

Definition 2.1.1. *[Borodin (2013), Chapter I, Section 3]. Let Σ be an arbitrary set. A stochastic process is a family*

$$X = \{X(t, \omega),\ t \in \Sigma\},$$

and for a fixed $\omega \in \Omega$ the map $t \to X_t$ is called a path. From now on, we omit the term ω and write X_t instead of $X(t, \omega)$. Moreover, in this thesis $\Sigma = \mathbb{R}_+$ and the only family of processes we work have càdlàg paths (right-continuous with left limits) almost surely. For simplicity, we say that such processes are càdlàg.

When we say that two process are equal, we mean by that the next definition.

Definition 2.1.2. *[Borodin (2013), Chapter I, Section 3] Two stochastic processes $X = \{X_t\}_{t\geq 0}$ and $Y = \{Y_t\}_{t\geq 0}$ defined in the same probability space*

(i) *are* stochastically equivalent *or* modifications *of each other if $\mathbf{P}(X_t = Y_t) = 1$, for all $t \geq 0$,*

(ii) *are* indistinguishable *or* equivalent *if $\mathbf{P}(X_t \neq Y_t$ for every $t \geq 0) = 0$.*

It is clear that (ii) implies (i) but in the case that the processes are càdlàg the conditions are equivalent.

We proceed to define a *filtered probability space*. Informally speaking it is a probability space with an associated sequence of σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\mathcal{F}_t$ represents the information up to time t.

Definition 2.1.3. *[Borodin (2013), Chapter I, Section 4] A family of σ-algebras $\mathbf{F} = \{\mathcal{F}_t,\ t \geq 0\}$ on $(\Omega, \mathcal{F})$ is called a* filtration *if*

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, \text{ for every } 0 \leq s \leq t.$$

Moreover we say that a filtration is right continuous *if for every $t \geq 0$:*

$$\mathcal{F}_t = \mathcal{F}_{t^+} := \bigcap_{h>0} \mathcal{F}_{t+h}.$$

The quadruple $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$ is called a filtered probability space. *It is said to satisfy the* usual conditions *if*

- *$\mathcal{F}$ is $\mathbf{P}$-complete,*
- *$\mathcal{F}_0$ contains all $\mathbf{P}$-null sets of $\mathcal{F}$ and*
- *$\mathbf{F}$ is right-continuous.*

In this thesis all filtrations satisfy the usual conditions.

In the next remark we mention *Feller processes* but due to not being used in this thesis we ommit its definition. However, we remark that Itô-diffusions and Lévy processes are part of this family of stochastic processes.

Remark 2.1.1. *Aside from the fact that the usual conditions saves us from working around with pathological cases, it is a natural property for the σ-algebras that we use in this thesis. To be more specific, the completed natural filtration of a* Feller process *satisfy the usual conditions.*

Definition 2.1.4. *[Borodin (2013), Chapter I, Section 4] A stochastic process $X = \{X_t\}_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$ is adapted to the filtration $\mathbf{F}$ if for every $t \geq 0$, X_t is $\mathcal{F}_t$-measurable. A stochastic process X is always adapted to its* natural filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{F}_t = \sigma(X_s,\ s \leq t)$. *We remark that a càdlàg adapted stochastic process is also* progressively measurable. *That is, for every $t \geq 0$, the map $(s, \omega) \to X(s, \omega)$ from $[0, t] \times \Omega$ to $\mathbb{R}$ is $\mathcal{B}([0, t]) \times \mathcal{F}_t$-measurable.*

Informally speaking, in the previous definition means that up to time t, a process can be described with the information up to time t.

Definition 2.1.5. *A random variable $\tau : \Omega \to [0, \infty]$ is a* stopping time *if the event $\{\tau \leq t\} \in \mathcal{F}_t$, for every t, $0 \leq t \leq \infty$. Moreover we define*

$$\mathcal{F}_\tau = \{A \in \mathcal{F}, A \cap \{\tau \leq t\} \in \mathcal{F}_t, \quad \textit{for all } t \geq 0\}.$$

Theorem 2.1.1. *[Protter (2005), Chapter I, Section 1, Theorems 3 and 4] Let $X = \{X_t\}_{t\geq 0}$ an adapted càdlàg stochastic process and A an open set then the following random variables are stopping times:*

$$\tau_A := \inf\{t : X_t \in A\}, \qquad \tau_{A^c} := \inf\{t : X_t \in A^c \textit{ or } X_t^- \in A^c\}.$$

From now on, unless specified we always assume a given filtered completed probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$.

2.1.2 Martingales and local martingales

Informally speaking, the purpose of this subsection is to formalize the concept of *working with information available at time t.*

Definition 2.1.6. *[Borodin (2013), Chapter I, Section 2] Let X be a random variable with $\mathbf{E}|X| < \infty$. The* conditional expectation $\mathbf{E}(X|\mathcal{Q})$ *of X given a σ-algebra $\mathcal{Q} \subset \mathcal{F}$ is the $\mathcal{Q}$-measurable random variable such that*

$$\int_B \mathbf{E}(X|Q)d\mathbf{P} = \int_B X d\mathbf{P},$$

for every $B \in \mathcal{Q}$. This random variable exists and is unique almost surely.

A martingale could be thought as an stochastic process X such that the best guess at time t of the value of the process X_t, based on only the information up to time $s \leq t$, is X_s (with some reasonable integrability conditions).

Definition 2.1.7. *[Borodin (2013), Chapter I, Section 5] An adapted stochastic process $X = \{X_t\}_{t\geq 0}$ is called* martingale *(respectively,* supermartingale, submartingale*) with respect to the filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$ if*

(i) $\mathbf{E}|X_t| < \infty$ *for all* $t \geq 0$.

(ii) $\mathbf{E}(X_t|\mathcal{F}_s) = X_s$ *(respectively , $E(X_t|\mathcal{F}_s) \leq X_s$, $E(X_t|\mathcal{F}_s) \geq X_s$)) a.s for every pair s, t such that $s \leq t$.*

Theorem 2.1.2 (Doob's Optional Sampling Theorem)**.** *Let X be a right-continuous martingale, such that there is a random variable $X_\infty \in L^1(\Omega)$ that satisfies $\mathbf{E}(X_\infty|\mathcal{F}_t) = X_t$ for all $t \geq 0$. Let $S \leq T$ a.s be two stopping times. Then X_S and X_T are integrable and $\mathbf{E}(X_T|\mathcal{F}_S) = X_S$ a.s*

The integrability conditions can be too restrictive. In some cases enough information can be obtained from a stochastic process if it is *locally* a martingale. Let us remark that we use the usual definition of *uniformly integrable* family of functions.

Definition 2.1.8. *[Protter (2005), Chapter I, Section 6] An adapted càdlàg process $X = \{X_t\}_{t\geq 0}$ is a* local martingale *if there exists a sequence of increasing stopping times τ_n with $\lim_{n\to\infty} \tau_n = \infty$ a.s. such that $X_{t\wedge\tau_n}\mathbf{1}_{\tau_n>0}$ is a uniformly integrable martingale for each n.*

Remark 2.1.2. *The condition $\tau_n > 0$ is considered in order to relax the integrability conditions of X_0. It can be proven that any martingale is a local martingale. The converse is not always true even if $\sup_t \mathbf{E}|X_t| < \infty$.*

2.2 Underlying processes

2.2.1 Introduction

The processes studied in this thesis are *controlled Itô-diffusions* and *controlled Lévy processes.* Before going into the definition of controls, let us start with the basics. We mention again that a filtered probability space is given.

Definition 2.2.1. *[Kyprianou(2006), Chapter I, Section 1] A real valued process $W = \{W_t\}_{t\geq 0}$ is said to be a* Brownian motion *if the following hold:*

(i) *The paths of W are $\mathbf{P}$-almost surely continuous.*
(ii) $\mathbf{P}(W_0 = 0) = 1$.
(iii) *For $0 \leq s \leq t$, $W_t - W_s$ is equal in distribution to W_{t-s}.*
(iv) *For $0 \leq s \leq t$, $W_t - W_s$ is independent of $\{W_u : u \leq s\}$.*
(v) *For each $t > 0$, W_t is equal in distribution to a normal random variable with variance t.*

Definition 2.2.2. *A process valued on the non negative integers $N = \{N_t\}_{t\geq 0}$ is said to be a Poisson process with intensity $\lambda > 0$ if the following hold:*

(i) *The paths of N are $\mathbf{P}$-almost surely càdlàg.*
(ii) $\mathbf{P}(N_0 = 0) = 1$.
(iii) *For $0 \leq s \leq t$, $N_t - N_s$ is equal in distribution to N_{t-s}.*
(iv) *For $0 \leq s \leq t$, $N_t - N_s$ is independent of $\{N_u : u \leq s\}$.*
(v) *For each $t > 0$, N_t is equal in distribution to a Poisson random variable with parameter λt.*

Definition 2.2.3. *[Kyprianou(2006), Chapter I, Section 1] Suppose that $N = \{N_t\}_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$ and that $\{\zeta_i : i \geq 1\}$ is an i.i.d. sequence of random variables (independent of N), a compound Poisson proces $X = \{X_t\}_{t\geq 0}$ is defined as*

$$X_t = \sum_{i=0}^{N_t} \zeta_i, \qquad \zeta_0 := 0.$$

2.2.2 Stochastic integration with respect to the Brownian motion

To define an Itô-diffusion, first we need to give a notion of an integral with respect to a Brownian motion. Although we use more general integrals and integrands in this thesis, the uncontrolled processes in our problems are simpler in nature.

Definition 2.2.4. *[Protter (2005), Chapter II, page 51] A process H is said to be* simple predictable *if H has a representation of the form*

$$H_t = H_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=1}^{n} H_i \mathbf{1}_{(T_i, T_{i+1}]}(t),$$

where $0 = T_1 \leq \cdots \leq T_{n+1} < \infty$ is a finite sequence of stopping times, $H_i \in \mathcal{F}_{T_i}$, $|H_i| < \infty$ a.s, $0 \leq i \leq n$. The collection of simple predictable processes is denoted by $\mathbf{S}$.

These are our stochastic *step functions* that allows to build stochastic integrals

Definition 2.2.5. *[Protter (2005), Chapter II, page 58] Let $W = \{W_t\}_{t\geq 0}$ be a Brownian motion and $H \in \mathbf{S}$ with the same notations as above, we define the stochastic integral of H with respect to W as:*

$$J_W(H) = (H \cdot W)_t = \int_0^t H_s dW_s := H_0 W_0 + \sum_{i=1}^{n} H_i (X_{T_{i+1}\wedge t} - X_{T_i \wedge t}).$$

Notice that we allow here $W_0 \neq 0$, that is we allow W to be a Brownian motion not starting at zero (a translated Brownian motion).

The next step is to extended J_W with an appropriate metric. We remark that naively using convergence a.s does not works in general (see [Protter (2005), Chapter I, Section 8]).

Definition 2.2.6. *A sequence of processes $\{H^n\}_{n\in\mathbb{N}}$ converges to a process H* uniformly on compacts in probability *(abbreviated* ucp*) if, for each $t > 0$, $\sup_{0\leq s\leq t} |H^n_s - H_s|$ converges to zero in probability.*

Theorem 2.2.1. *[Protter (2005), Chapter I, Theorem 10 and Chapter II, Theorem 11] The space $\mathbf{S}$ is dense in $\mathbb{L}$ with the* ucp *topology, where $\mathbb{L}$ denotes the space of left-continuous with right limits stochastic processes. Moreover, the map $J_W : \mathbf{S}_{\text{ucp}} \to \mathbb{D}_{\text{ucp}}$, where $\mathbb{D}_{\text{ucp}}$ are the càdlàg processes embedded with the* ucp *topology is a continuous map.*

With these results we can define the integral in a wider set of stochastic processes.

Definition 2.2.7. *Denoting $\mathbb{L}_{\text{ucp}}$ the set of left-continuous with right limits stochastic processes, we define the map $J_W : \mathbb{L}_{\text{ucp}} \to \mathbb{D}_{ucp}$ as the extension of J_W defined in 2.2.5 and call it* stochastic integral with respect to Brownian motion.

The integrands can be in fact be in a more general family, we will return to this topic later, we refer to the books [Borodin (2013), Jacod and Shiryaev (2003), Protter (2005)].

Remark 2.2.1. *The* usual *properties of integrals hold (see [Protter (2005), Chapter II, Section 5] or [Borodin (2013), pages 88 and 89]).*

2.2.3 Itô-diffusion

An Itô-diffusion is a solution to a specific type of *stochastic differential equation* driven by a Brownian motion. In other words, it is in the class of the solutions of an integral equation with respect to dt and dW_t.

Definition 2.2.8. *A process X_t, $X_0 = \eta$, is said to be a* strong solution *of the* stochastic differential equation *(abbreviated SDE) up to the stopping time τ*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,\ X_0 = \eta, \tag{2.1}$$

if X is continuous $\mathcal{F}_t$-adapted process such that almost surely for all $t \leq \tau$

$$X_t - X_0 = \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,\ X_0 = \eta.$$

From now on, we denote $\mathbf{P}_x, \mathbf{E}_x$ the associated probability measure and expected value respectively when the process satisfies $X_0 = x$.

Remark 2.2.2. *This definition suggests the existence of a* weak solution, *such a solution exists when the probability space needs to be enlarged. Heuristically, as we do not want to modify our* information, *in this thesis we only work with* strong solutions.

Theorem 2.2.2. *[Borodin (2013), Chapter II, Theorem 7.1] Fix $T \in [0, \infty]$. Suppose that functions μ and σ satisfy Lipschitz conditions up to time T, both grow at most linearly up to time T and $\mathbf{E}\eta^2 < \infty$. Then there exists a unique strong solution of 2.1 up to T satisfying the conditions*

$$\sup_{0 \leq t \leq T} \mathbf{E}X^2(t) < \infty.$$

The Lipschitz Hypothesis can be weakened, see [Protter (2005), Chapter V], for a more exhaustive study.

Theorem 2.2.3. *[Protter (2005), Chapter VI, Theorem 38] Assume μ and σ are locally Lipschitz. Then there is a function $\zeta(x, \omega) : \mathbb{R} \times \Omega \to [0, \infty]$ such that for each x, $\zeta(x, \cdot)$ is a stopping time and there exists a unique strong solution of 2.1 up to $\zeta(x, \cdot)$ with $\limsup_{t \to \zeta(x, \cdot)} |X_t| = \infty$ almost surely on $\{\zeta < \infty\}$ and X has continuous paths on $[0, \zeta(x, \cdot))$.*

The function ζ is called *explosion time.*

Definition 2.2.9. *With the notations of Definition 2.1 and under the hypothesis of 2.2.3, if μ and σ do not depend on t and $\sigma > 0$, we say that it is a* strong solution *up to its explosion time is an Itô-Diffusion (sometimes called homogenous Itô-Diffusion).*

Remark 2.2.3. *The condition $\sigma > 0$, makes the diffusion* regular, *without entering into technical details, for every arbitrary point y, the $\mathbf{P}_x$ probability of reaching that point in finite time is not zero for every $x \in \mathbb{R}$ (see [Rogers and Williams (2000), Chapter V, Section 47, Remark* (ii)*]). This is fundamental to obtain ergodic properties that will be used in 3 and 4.*

We proceed to define the *density of the scale* and *the speed measure*. These notions will have an important role in 3 and 4, ([Borodin and Salminen(2002), Chapter II, Definition 4] and [Rogers and Williams (2000), Chapter V, section 45]).

Definition 2.2.10. *The density of the scale function $S(x)$ w.r.t the Lebesgue measure as*

$$S'(x) = \exp\left(-\int^{x} \frac{2\mu(u)}{\sigma^2(u)}du\right),$$

and the density of the speed measure $m(x)$ w.r.t the Lebesgue measure as

$$m'(x) = \frac{2}{\sigma^2(x)S'(x)}.$$

Remark 2.2.4. *The scale can also be defined as*

$$\mathbf{P}_x(\gamma_{[b,\infty)} - \gamma_{(-\infty,a]}) = \frac{S(x)-S(a)}{S(b)-S(a)}, \ a < x < b,$$

where, γ_C is defined as the first entry to the set C.

Remark 2.2.5. *The speed measure also has a more intuitive, yet analytically less clear in our case, way to be defined. However, it requires us to define* local times*, so we simply remark that when $s(x) = x$ for all x, the bigger is the number $m([a, b])$, the slower the diffusion moves in that interval.*

2.2.4 Lévy processes

The other family of processes that we study in this thesis are the Lévy processes. In certain sense, due to their properties, the arguments we make in this thesis in the optimization problems when the underlying process is Lévy are more probabilistic in nature that the ones put forward when the process is an Itô-diffusion (which are more analytical). As usual a probability space under the usual hypothesis is given.

Definition 2.2.11. *[Kyprianou(2006), Chapter I, Definition 1.1] A process $X = \{X_t\}_{t\geq 0}$ is a said to be a Lévy process if it possesses the following properties:*

(i) *The paths of X are $\mathbf{P}$ almost surely right-continuous with left limits.*
(ii) $\mathbf{P}(X_0 = 0) = 1$.
(iii) *For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution in X_{t-s}.*
(iv) *For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.*

From now on, we denote $\mathbf{P}_x$, $\mathbf{E}_x$ the associated probability measure and expected value respectively when a process satisfies $X_t - x$ is a Lévy process and call it a Lévy process starting at x. As expected, the sum of independent Lévy processes is a Lévy process. Moreover, Lévy processes are strong markov processes, in fact they satisfy the stronger following property.

Theorem 2.2.4. *[Kyprianou(2006), Chapter III, Theorem 3.1] Suppose that τ is a stopping time. Define on $\{\tau < \infty\}$ the process $Y = \{Y_t\}_{t\geq 0}$ where $Y_t = X_{\tau+t} - X_\tau$. Then on the event $\{\tau < \infty\}$ the process Y is independent of $\mathcal{F}_t$ and has the same law as X and hence in particular is a Lévy process.*

An important property of the Lévy processes (see for example, [Kyprianou(2006), Chapter I, equation (1.2)]) is that for each $t > 0$:

$$\phi_t(z) = \log\left(\mathbf{E}(e^{zX_t})\right) = t\log\left(\mathbf{E}(e^{zX_1})\right) = (\phi_1(z))^t, \qquad z = i\theta \in i\mathbb{R},$$

It can be proven that the function ϕ_1 (which we denote ϕ from now on) characterizes the law of the process. From now on, we assume that the $\mathbf{E}|X_1| < \infty$ (which implies $\mathbf{E}|X_t| < \infty$ for all $t \geq 0$). We remark that the Brownian motion has finite moments. Moreover a compound Poisson process has finite moments iff its jumps have finite moments. The Lévy-Khintchine formula characterizes the law of the process, stating

$$\phi(z) = \log\left(\mathbf{E}(e^{zX_1})\right), \qquad z = i\theta \in i\mathbb{R},$$

with

$$\phi(z) = \frac{\sigma^2}{2}z^2 + z\mu + \int_{\mathbb{R}} (e^{zy} - 1 - zy)\,\Pi(dy),$$

where $\mu = \mathbf{E}(X_1) \in \mathbb{R}$, $\sigma \geq 0$ and $\Pi(dy)$ is a non-negative measure (the *jump measure*) that satisfies in our case $\int_{\mathbb{R}}(y^2 \wedge |y|)\Pi(dy) < \infty$. This claim is valid (although the formula is simplified in this case) when the process has no finite expected value. The proof of this is in plenty of books, we refer to [Kyprianou(2006), Chapter I, Theorems 42 and 43], for a probabilistic approach and [Sato (1999), Chapter II], for a more analytic approach. We remark that the converse is also true. To be more specific, if there is a function ϕ satisfying the conditions above, then there is a Lévy process X such that $\phi(z) = \log\left(\mathbf{E}(e^{zX_1})\right)$, $z = i\theta$. The Lévy-Khintchine formula also allows to describe any Lévy process as the sum of a Brownian motion with drift, a compound Poisson process and a *series of compensated compound Poisson processes*. To understand the last process we need some definitions first.

Definition 2.2.12. *[Kyprianou(2006), Chapter II, Definition 2.3], Let $(S, \mathcal{S}, \eta)$ an arbitrary σ-finite measure space. Let $N : \mathcal{S} \to \{0, 1 \dots\} \cup \{\infty\}$ in such a way that the family $\{N(A) :$*

$A \in \mathcal{S}\}$ *are random variables defined on the probability space* $(\Omega, \mathcal{F}, \mathbf{P})$. *Then* N *is called a Poisson random measure on* $(S, \mathcal{S}, \eta)$ *if*

(i) *for mutually disjoint* $A_1, \ldots, A_n$ *in* $\mathcal{S}$, *the variables* $N(A_1), \ldots, N(A_n)$ *are independent.*

(ii) *for each* $A \in \mathcal{S}, N(A)$ *is Poisson distributed with parameter* $\eta(A)$ *(here we allow* $0 \leq \eta(A) \leq \infty$.

(iii) $\mathbf{P}$*-almost surely* N *is a measure.*

We remark, that in general for a $(S, \mathcal{S}, \eta)$ as above there exists a Poisson random measure (see [Kyprianou(2006), Chapter II, Theorem 2.4]).

Lemma 2.2.5. *[Kyprianou(2006), Chapter II, Lemma 2.8 and 2.9] Suppose that* N *is a Poisson random measure on* $([0, \infty) \times \mathbb{R}, \mathcal{B}[0, \infty) \times \mathcal{B}(\mathbb{R}), dt \times \Pi(dx))$ *where* Π *is a measure concentrated on* $\mathbb{R} - \{0\}$ *and* $B \in \mathcal{B}(\mathbb{R})$ *such that* $0 < \Pi(B) < \infty$. *Then*

(i)
$$X_t := \int_{[0,t]} \int_B x N(ds \times dx), \; t \geq 0$$

is a compound Poisson process with arrival rate $\Pi(B)$ *and jump distribution* $\Pi(B)^{-1}\Pi(dx)|_B$

(ii)
$$M_t^B := X_t - t \int_B x \Pi(dx), \; t \geq 0$$

is a $\mathbf{P}$*-martingale with respect to the filtration*

$$\sigma(N(A) : A \in \mathcal{B}[0, t] \times \mathcal{B}(\mathbb{R})), \; t \geq 0. \tag{2.2}$$

We call such a martingale a compensated compound Poisson process.

Theorem 2.2.6. *[Kyprianou(2006), Chapter II, Theorem 2.8] With the notations of the previous lemma, assume* $\int_{(-1,1)} x^2 \Pi(dx) < \infty$ *and for* $\epsilon > 0$ *define* $B_\epsilon := (-1, 1) - (-\epsilon, \epsilon)$. *Let* $M_t^{B_\epsilon}$ *be the martingales defined in the previous lemma for the completation of the filtration* (2.2). *Then, there exists a Lévy process* $M = \{M_t\}_{t \geq 0}$, *which also is a martingale with countable number of jumps to which* $\{\{M_t^{B_\epsilon}\}_{t \geq 0}\}_{\epsilon > 0}$ *has a subsequence* $\{\{M_t^{B_{\epsilon_n}}\}_{t \geq 0}\}_{n \in \mathbb{N}}$, $\epsilon_n \to 0$ *when* $n \to \infty$ *converges in* ucp *to* M.

From now on, the measure $\tilde{N}(ds, dy) = N(ds, dy) - ds\Pi(dy)$ will be called compensated Poisson random measure and the martingale M_t defined in the previous lemma will be written as

$$M_t = \int_{[0,t] \times \mathbb{R}} y \, \tilde{N}(ds, dy).$$

The next theorem holds for every Lévy process, but as we only work with processes with first moments, we choose to give a simplified version. For references, see for example [Borodin (2013), Sato (1999), Protter (2005), Jacod and Shiryaev (2003)].

Theorem 2.2.7 (Lévy decomposition Theorem). *Every Lévy process with finite moments, can be expressed as*

$$X_t = X_0 + \mu t + \sigma W_t + \int_{[0,t]\times\mathbb{R}} y\,\tilde{N}(ds, dy).$$

with W and $\tilde{N}$ independent processes.

Basically, the Theorem boils down to:

- The big jumps give a compound Poisson process.
- The small jumps are compensated and M_t is obtained.
- What is left is a continuous Lévy process. The only continuous Lévy process is the Brownian motion with drift.

From the decomposition, the following lemma is obtained

Lemma 2.2.8. *[Kyprianou(2006), Chapter II, Lemma 2.12] A Lévy process with a Lévy-Khintchine exponent corresponding to the triple (μ, σ, Π) has paths of bounded variation if and only if*

$$\sigma = 0, \qquad \int_{\mathbb{R}} (1 \wedge |x|)\Pi(dx) < \infty.$$

In that case we denote

$$S_t^+ := \max(\mu, 0)t + \int_0^t \int_{x\geq 0} xN(ds \times dx), \quad S_t^- := -\min(\mu, 0)t - \int_0^t \int_{x\leq 0} xN(ds \times dx).$$

These processes are independent subordinators (increasing Lévy processes).

2.3 Stochastic calculus

In this section we give a brief resume of the stochastic calculus tools we are going to use. First we need to define *good integrators*.

2.3.1 Semimartingales and stochastic integration

We use Definition 2.2.4 for $\mathbf{S}$. Furthermore we denote $\mathbf{S}_u$ when $\mathbf{S}$ is endowed with the uniform topology in (t, ω) and $\mathbf{L}^0$ the space of finite random variables topologized by convergence in probability.

Definition 2.3.1. *A càdlàg, adapted stochastic process $X = \{X_t\}_{t\geq 0}$ is a semimartingale if for every $T \geq 0$, the map:*

$$I_{X^T}(H) = H_0 X_0^T + \sum_{i=1}^{n} H_i (X_{T_{i+1}\wedge T} - X_{T_i \wedge T})$$

is continuous (H as in 2.2.4).

This definition clarifies why a semimartingale is a *good integrator*, however, it is quite abstract. An equivalent definition is that a semimartingale is a process that can be decomposed as $X_t = M_0 + M_t + A_t$ with M_t a local martingale and A_t a cádlag finite variation process. Lévy processes and Itô-diffusions are examples of semimartingales.

2.3.2 Stochastic integration

Integration with respect to a martingale can be defined for *predictable processes*, since in this thesis it is not necessary, we simply extend it for left-continuous with right limits processes.

Theorem 2.3.1. *Theorem 2.2.1 is valid if we replace J_W by J_X, for any $X = \{X_t\}_{t\geq 0}$ semimartingale.*

We then define the stochastic integral with respect to a semimartingale $X = \{X_t\}_{t\geq 0}$ as in Definition 2.2.7 but with J_X instead of J_W. Again, the expected linear properties hold.

A integral with respect to a martingale does not needs to be a martingale, however the following lemma holds.

Lemma 2.3.2. *If $Z = \{Z_t\}_{t\geq 0}$ is a semimartingale and $X = \{X_t\}_t$ is a local martingale then*

(i) $I_t = \int_0^t Z_{s^-} dX_s$ is a local martingale.

(ii) If X is a Lévy process and Z is bounded then I_t is a martingale.

The first statement can be found for example in [Protter (2005), Chapter IV, Section 2, Theorem 29]. For the second one, we prove it in the appendix, Lemma A.2.1 as it will be used in Chapter 5.

Under this framework a *stochastic differential equation* is of the form

$$X_t - X_0 = \int_0^t f(s, X_{s^-}) dZ_s,$$

where Z is a semimartingale. As said before, we are only interested in *strong solutions*. We do not list general results here of existence and uniqueness (again see for example [Protter (2005)]) as the only classes of SDE we study will be treated with more detail.
Although they are not the focus in this thesis, we define Itô-jump-diffusions as they are referenced many times.

Definition 2.3.2. *[Øksendal & Sulem, (2005), Chapter I, Section 1.3] The following SDE in $\mathbb{R}^n$ (here the SDE is defined as a vector) is called Itô-Jump-diffusion:*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbb{R}^n} \gamma(X_{t^-}, y)\tilde{N}(ds \times dy), \qquad X(0) = x \in \mathbb{R}^n.$$

where $\mu : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$, $\gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $W = \{W_t\}_{t\geq 0}$ is a multidimensional Brownian motion. To simplify the notation, We say X is an Itô-jump-diffusion with parameters $(\mu, \sigma, \gamma, \tilde{N})$

2.3.3 Itô formula

As explained in the introduction, we are interested in controlled Lévy processes and Itô-diffusions by processes of bounded variation. As it is our intention to make this thesis reasonably self-contained, we write several versions of the Itô formula to avoid dealing with the theory of semimartingale compensation and quadratic variation. There are many books where the Itô formula is proven, we put the reference of the one we transcribed the text. The first statement is [Protter (2005), Chapter II, Theorem 31]. The second and third statements are a particular case of [Protter (2005), Chapter II, Theorem 32].

Theorem 2.3.3. *Let V be a process of finite variation with right-continuous paths. Suppose $f \in C^1(\mathbb{R})$. Then $f(V) := \{f(V_t)\}_{t\geq 0}$ is a finite variation process and*

$$f(V_t) - f(V_0) = \int_{0^+}^t f'(V_{s^-})dV_s + \sum_{0<s\leq t} \left(f(V_s) - f(V_{s^-}) - f'(V_{s^-}) \triangle V_s\right).$$

Theorem 2.3.4. *Assume μ and σ are continuous functions. $U = \{U_t\}_{t\geq 0}$, $D = \{D_t\}_{t\geq 0}$ are a pair of cádlag processes with finite first moments for each $t \geq 0$ and the SDE*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dU_t - dD_t, \quad X_0 = x,$$

has a unique strong solution with no finite explosion time. Then, if $f \in C^2(\mathbb{R})$ is a function with first derivate constant outside an interval:

$$
\begin{aligned}
f(X_t) - f(X_0) = &\int_{0+}^{t} f'(X_{s^-})(\mu(X_{s^-})ds + dU_s - dD_s) \\
&+ \frac{1}{2}\int_{0+}^{t} \sigma^2(X_{s^-})f''(X_{s^-})ds + \sum_{0<s\leq t} (f(X_s) - f(X_{s^-}) - f'(X_{s^-}) \triangle X_{s^-}) \,.
\end{aligned}
$$

Theorem 2.3.5. *Assume $X = \{X_t\}_{t\geq 0}$ is a Lévy process with finite first moment and unbounded variation. $U = \{U_t\}_{t\geq 0}, D = \{D_t\}_{t\geq 0}$ are a pair of cádlag processes with finite first moment for each $t \geq 0$. Then, if $f \in C^2(\mathbb{R})$ is a function with first derivate constant outside an interval:*

$$
\begin{aligned}
f(X_t) - f(X_0) = &\int_{0+}^{t} f'(X_{s^-})dX_s + \frac{\sigma^2}{2}\int_{0+}^{t} f''(X_{s^-})ds \\
&+ \sum_{0<s\leq t} (f(X_s) - f(X_{s^-}) - f'(X_{s^-}) \triangle X_s) \,.
\end{aligned}
$$

In the following chapters the reader will observe that the lower limit in the integrals is 0 instead of 0^+, in fact this varies in the literature but it is just a problem in notation. To be more clear, we define $X_{0^-} := 0$ and notice that in the processes we study there is no jump at $t = 0$.

2.4 Excessive functions and the infinitesimal generator

In our optimizations problems, we need to formalize the concept of a function where *it is not convenient to leave the underlying process uncontrolled in a certain region.* For the following definitions (i) and (ii) see [Peskir and Shiryaev (2006), Chapter I, Section 4] and [Peskir(2009), Section 2] respectively.

Definition 2.4.1. *Consider $V : \mathbb{R} \to \mathbb{R}_+$ a measurable function. We say that*

(i) *V is* $\mathbf{P}$-excessive *or* superharmonic *(*$\mathbf{P}$- subharmonic*) if*

$$
\mathbf{E}_x(V(X_t)) \leq \ (\geq)\ V(x), \ \textit{for all } x \in \mathbb{R}.
$$

(ii) *Given a Borel set C, we say V is* $\mathbf{P}$-excessive *in C or* superharmonic *in C (*$\mathbf{P}$ -subarmonic *in C) if*

$$
\mathbf{E}_x(V(X_{t\wedge\gamma_{C^c}})) \leq \ (\geq)\ V(x), \ \textit{for all } x \in \mathbb{R},
$$

where

$$\gamma_{C^c} = \inf_{t \geq 0}\{X_t \notin C\}.$$

Speaking informally, excessiveness means that if we let the process run up to time t, then in mean, the value obtained will be smaller than the initial value. Moreover excessivenes in C means the same, with the precaution that we must stop the process when we leave C.
In some cases, the objective is not to compare the process at the starting time with the future, but to compare it by taking into account that there is a *depreciation* in value over time and a *cost over observations.* In other words, we are interested in inequalities (and equalities) of the type:

$$\mathbf{E}_x(e^{-\epsilon(t\wedge\gamma_{C^c})}V(X_{t\wedge\gamma_{C^c}})) + \int_0^{t\wedge(\gamma_{C^c})} c(X_s, s)ds \leq \ (\geq) \ V(x), \text{ for all } x \in \mathbb{R}. \tag{2.3}$$

For a reference, see for example, [Peskir and Shiryaev (2006), Chapter III, Section 7].
As mentioned earlier, the study of the operator $(t, V) \to \mathbf{E}_x(V(X_t))$ defined for a *nice* set of functions is essential for solving the problems posed in this thesis. This operator is an example of *contraction semigroup.* For the next definition and theorem, we refer to [Borodin (2013), Chapter IV], [Sato (1999), Chapter VI] and [Dynkin, E. B. (1965), Chapter II].

Definition 2.4.2. *Given a strong Markov process $X = \{X_t\}_{t\geq 0}$. We define the operator $\mathcal{L}$, called infinitesimal generator, as:*

$$\mathcal{L}_X u(x) := \lim_{t\to 0} \frac{\mathbf{E}_x u(X_t) - u(x)}{t}, \tag{2.4}$$

whose domain are the measurable functions such that the limit u exists for every $x \in \mathbb{R}$. We call such set domain of the infinitesimal generator*. As a remark, the* infinitesimal version *of equation* (2.3) *is of the form:*

$$\mathcal{L}_x u(x) - \epsilon u(x) + c(x) \leq \ (\geq) \ 0.$$

Theorem 2.4.1. *If $X = \{X_t\}_{t\geq 0}$ is an Itô-difussion or a Lévy process then the set $C_0^2(\mathbb{R})$ of twice continuously differentiable functions with compact support is in the domain of the infinitesimal generator and for every $u \in C_0^2(\mathbb{R})$:*

(i) *if the process is an Itô-diffusion (with the usual notations) then*

$$\mathcal{L}_X u(x) = \mu(x)u'(x) + \frac{\sigma^2(x)}{2}u''(x),$$

(ii) *if the process is Lévy (with the usual notations) then*

$$\mathcal{L}_X u(x) = \mu u'(x) + \frac{\sigma^2}{2} u''(x) + \int_{\mathbb{R}} \left(u(x+y) - u(x) - u'(x) y \mathbf{1}_{|y| \leq 1} \right) d\Pi(y).$$

As commented in the definition of an excessive function on a set, it will be helpful if we could be able to obtain properties of the operator $x \to \mathbf{E}_x(X_{t \wedge \gamma_{C^c}})$. This is where the *characteristic operator* comes into play. That is, instead of taking t deterministic in the denominator and the process X_t in the nominator of the limit (2.4) we take $\mathbf{E}_x(\gamma_{C^c})$ and $X_{\gamma_{C^c}}$ respectively. Where γ_{C^c} is the first exit of a neighborhood of x and the limit is taken by shrinking the neighborhoods. In some cases, both generators coincide, see for example [Dynkin, E. B. (1965), Chapter V, Theorem 5.2].

To conclude this section, we explain what we mean when we denote *Hamilton-Jacobi-Bellman* equation (abbreviated HJB equation). Following [Festa. et. al. (2017), Section 1.2], we do not give a precise definition. The next discussion is based on [Yong and Zhou(1999), Chapter IV, Setion 1]. Each departing state X_0 of our underlying processes determines a different stochastic optimization problem. The relationship between these problems is established through second order differential equations when the process is an Itô-diffusion or a second order integro-differential-equation when the process is Lévy. Due to the nature of our problems, informally speaking, we can define in this thesis a HJB equation as a finite number of inequalities where the infinitesimal generator appears and characterizes the solution of our control problem. The usual limitation of this approach is that too much smoothness is required for these equations to make sense. Nevertheless in this thesis we do not have this problem as the solutions are smooth enough.

2.5 Reflection on intervals

In this section we give the minimal theoretical framework necessary to work with a family of processes where the optimal controls of our optimization problems are found. Due to the nature of our problems, we will prove in this thesis that the best course of action is to choose an adequate couple of barriers $a \leq b$ (the inequality always strict if the process has unbounded variation) and *prevent the process from leaving the set* $[a, b]$ *with the minimal possible push.* One can get the idea of what we mean when the process is a compound Poisson process but the definition is less intuitive when the process, for example, has unbounded variation.

Remark 2.5.1. *The reader may notice that in the case where there is only one barrier we are* *talking about the* local time.

2.5.1 Deterministic case

In this subsection $\mathbb{D}[0,\infty)$ denotes the set of càdlàg functions mapping $[0,\infty)$ into $\mathbb{R}$. The next definition is adapted from [Kruk et. al. (2008)].

Definition 2.5.1. *Let $a < b$ be a pair of real numbers. The* double Skorkhod map $\Gamma_{a,b}$ *is the mapping from $\mathbb{D}[0,\infty)$ into itself such that for $\rho \in \mathbb{D}[0,\infty)$, $\Gamma_{a,b}(\rho)$ takes values in $[a,b]$ and has the decomposition.*

$$\Gamma_{a,b}(\rho) = \rho + U^{a,b} - D^{a,b},$$

where $U^{a,b}, D^{a,b} \in \mathbb{D}[0,\infty)$ are non-decreasing and satisfy

$$\int_0^\infty (\Gamma_{a,b}(\rho)(t) - a)dU_t^{a,b} = 0, \qquad \int_0^\infty (b - \Gamma_{a,b}(\rho)(t))dD_t^{a,b} = 0. \tag{2.5}$$

Moreover, in $t = 0$ the functions $U^{a,b}, D^{a,b}$ project $\rho(0)$ to the closest point in $[a,b]$.

As one might expect, *Skorokhod map* with a one-sided reflection was posed before the *double Skorokhod map* (see [Skorokhod (1961)]). We do not formalize this problem, as it is never used in the thesis. Its definition is similar to the one above but the controlled function Γ_a takes values in $[a,\infty)$, there is no function $D^{a,b}$ and in (2.5) there is only one equality (the one in the left). Its explicit representation is

$$\Gamma_a(\rho)(t) = \rho(t) + \sup_{s\in[0,t]} (-\rho(s) + a)^+. \tag{2.6}$$

In [Kruk et. al. (2007)], the authors showed that the *double Skorkhod map* is well defined, in the sense that for every $\rho \in \mathbb{D}[0,\infty)$ there is a unique pair $U^{a,b}, D^{a,b}$ as the definition above. Moreover it can be expressed in the following way:

Theorem 2.5.1. *Given $b > 0$, the* double Skorkhod map $\Gamma_{0,b}$ *exists and is well defined. Moreover:*

$$\Gamma_{0,b}(t) = \Lambda_b \circ \Gamma_0(t),$$

where Γ_0 is defined as in (2.6) and

$$\Lambda_b(\varphi)(t) = \varphi(t) - \sup_{s\in[0,t]} \left((\varphi(s) - b)^+ \wedge \inf_{u\in[s,t]} \varphi(u) \right).$$

In [Kruk et. al. (2008), Theorem 2.1], the authors proved that the *double Skorkhod map* can be expressed as:

$$\Gamma_{0,b}(\rho)(t) = \rho(t) - \max\left((\rho(0)-b)^+ \wedge \inf_{u\in[0,t]} \rho(u)\ ,\ \sup_{s\in[0,t]} \left((\rho(s)-b)^+ \wedge \inf_{u\in[s,t]} \rho(u) \right) \right).$$

Remark 2.5.2. *This approach can be done path by path for Lévy processes due to its spatially homogeneity.*

2.5.2 Stochastic general framework

As it will be mentioned in some historical remarks, we proceed to define the reflection of a stochastic process in an interval $[a, b]$. The underlying process is the strong solution of the SDE:

$$dX_t = dH_t + \int_0^t f(X_{s^-})dZ_s, \tag{2.7}$$

where H_t is an adapted process, Z_t is an adapted semimartingale starting at zero and f is a measurable function.

Definition 2.5.2. *We say that $X^{a,b}$ is the* reflected process *of X within the barriers $a < b$ and $(U^{a,b}, D^{a,b}) = (\{U_t^{a,b}\}_{t\geq 0}, \{D_t^{a,b}\}_{t\geq 0})$ are its* reflecting controls *if*

(i) *the process $X_t^{a,b} \in [a, b]$ for all $t \geq 0$ and is a strong solution of the SDE*

$$dX_t^{a,b} = dH_t + \int_0^t f(X_{s^-}^{a,b})dZ_s + dU_t^{a,b} - dD_t^{a,b},$$

(ii) *the processes U, D are increasing with $U_0^{a,b} = (a-x)^+$ and $D_0^{a,b} = (x-b)^+$ and*

$$\int_0^\infty (X_t^{a,b} - a)dU_t^{a,b} = 0, \quad \int_0^\infty (b - X_t^{a,b})dD_t^{a,b} = 0.$$

In the paper [Słomíski (1993)] the author gives conditions for the existence and uniqueness of *reflecting controls* given a process (2.7) (in a multidimensional setting).

2.5.3 Itô-diffusion

When the underlying process is an Itô-diffusion, in [Lions and Sznitman (1984)] and then [Saisho (1987)] for a more general domain, the authors gave conditions for the existence and uniqueness of a *reflected process* in a domain $\mathcal{D}$. We proceed to state the result for the particular case $\mathcal{D} = [a, b]$ with $a < b$. From the fact that the next Theorem is a simplification of [Lions and Sznitman (1984), Theorem 3.1], we prove it in the appendix, Theorem A.1.2.

Theorem 2.5.2. *Assume that X is an Itô-diffusion, μ, σ are locally Lipschitz and $a < b$. Then there is a unique pair $(U^{a,b}, D^{a,b}) = (\{U_t^{a,b}\}_{t\geq 0}, \{D_t^{a,b}\}_{t\geq 0})$ satisfying* (i) and (ii) of Definition 2.5.2.

Finally, we postulate an ergodic theorem for real Itô-diffusions which will be useful in 3. For a proof, see for example [Rogers and Williams (2000), Chapter V, Theorem 53.1].

Theorem 2.5.3. *Suppose X is an Itô-diffusion and f a locally bounded measurable non negative function, then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(X_s^{a,b}) ds = \int_a^b f(x) m'(x) dx, \ \ a.s.$$

This formula is our main weapon to pass from ergodic probabilistic problems to analytic ones. For Lévy process we need to use different techniques.

2.5.4 Lévy processes

Due to the fact that it is spatially homogeneous, in Definition 2.5.2, H_t can be taken as the null process, f as constant 1 and Z_t as a Lévy process. Therefore, $U^{a,b}, D^{a,b}$ can be obtained as in the deterministic case path by path. Some useful time dependant and ergodic properties from [Andersen et al. (2015)] are used and explained in detail in Chapter 5, Section 5.6. As a final remark, when the process has bounded variation it makes sense to define $(U^{0,0}, D^{0,0})$ and there is a convergence, in L^1 in the barriers, this will be proved in Chapter 5, Section 5.5.2.

2.6 Regenerative theory

We give some results of *renewal theory* that will be used for Lévy processes in Chapter 6. Basically we are interested in processes that can be split into i.i.d *cycles* to study their ergodic properties. In this thesis the *cycle* consists of the first time the lower barrier a is reached after reaching the upper barrier b.

Definition 2.6.1. *[Asmussen S. (2008), Chapter V, Section 1], Let $0 \leq S_0 \leq S_1 < S_2 < \dots$ be* the times of occurrence of some phenomenon *(in this thesis a stopping time) and $Y_n = S_n - S_{n-1}$, $Y_0 = S_0$. Then $\{S_n\}_{n\in\mathbb{N}}$ (with the zero included) is called renewal process if $Y_0, Y_1, \dots$ are independent and $Y_1, Y_2, \dots$ (but not necessarily Y_0) have the same distribution.*

The following is an example of renewal process that will be used in Chapter 6 and it is studied in Lemma A.2.3.

Lemma 2.6.1. *Let $X = \{X_t\}_{t\geq 0}$ be a Lévy process with finite mean such that X is not trivial, is not a subordinator, nor the opposite of a subordinator. Let $\{\tau_n\}_n$ be defined as:*

$$\tau_0 = \inf\{t \geq 0, X_t^{0,b} = 0, \sup_{0\leq s\leq t} X_s^{0,b} = b\},$$
$$\tau_{n+1} = \inf\{t \geq \tau_n,\ X_t^{0,b} = 0, \sup_{\tau_n\leq s\leq t} X_s^{0,b} = b\}.$$

Then $\{\tau_n\}_{n\in\mathbb{N}}$ is a renewal process and $\mathbf{E}(\tau_n) < \infty$ for every $n \in \mathbb{N}$.

Definition 2.6.2. *[Asmussen S. (2008), Chapter VI, Section 3]*

A real–valued process $\{Z_t\}$ is called cumulative if $Z_0 = 0$ and there exists a renewal process $\{S_n\}$ such that for any n, $\{Z_{S_n+t} - Z_{S_n}\}$ is independent of $S_0,\ S_1, \ldots, S_n$ and $\{Z_t\}_{t<S_n}$, and for every $t \geq 0,\ n, m \in \mathbb{N}$: $Z_{S_n+t} - Z_{S_n} = Z_{S_m+t} - Z_{S_m}$ in law.

Theorem 2.6.2. *[Asmussen S. (2008), Chapter VI, Theorem 3.1] Suppose $\{S_n\}$ is a renewal process and Z_t is an accumulative process. Moreover assume $S_0 = 0$, $\mathbf{E}(S_2-S_1) < \infty$, $\mathbf{E}|Z_{S_1}| < \infty$. Then*

$$\lim_{t\to\infty} \frac{Z_t}{t} = \frac{\mathbf{E}(Z_{S_1})}{\mathbf{E}(S_2 - S_1)} \ \ a.s \quad \textit{if and only if}\ \mathbf{E}\left(\max_{0\leq t\leq S_1} |Z_t|\right) < \infty.$$

2.7 Stochastic singular control problems

In this section we define the main problems that we are going to treat in chapters 3 and 5. In order to see the analysis of some selected references, see the corresponding chapters. We are concerned with finding the best policy for a long time average stochastic optimization singular control problem. We treat the ergodic problem when the underlying process is an Itô-diffusion and a Lévy process. We also treat the discounted problem when the underlying process is Lévy and study the abelian limit. As usual, we work with a standard filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P}_x)$ when the underlying stochastic process $X = \{X_t\}_{t\geq 0}$ is a real Itô-jump-diffusion with parameters $(\alpha, \sigma, \gamma, \tilde{N})$.

Definition 2.7.1. *An* admissible control *is a pair of non-negative $\{\mathcal{F}_t\}$-adapted processes $\eta = (U = \{U_t\}_{t\geq 0}, D = \{D_t\}_{t\geq 0})$ such that:*

(i) *Each process U, D is right-continuous and non-decreasing almost surely.*

(ii) *For each $t \geq 0$ the random variables U_t and D_t have finite expectation.*

(iii) *For every $x \in \mathbb{R}$, the stochastic differential equation*

$$dX_t^{U,D} = \mu(X_t^{U,D})dt + \sigma(X_t^{U,D})dB_t + \int_{\mathbb{R}^n} \gamma(X_{t^-}^{U,D}, y)\tilde{N}(ds \times dy) + dU_t - dD_t,$$
$$X^{U,D}(0) = x + U_0 - D_0 \in \mathbb{R}.$$

has a unique strong solution with no explosion in finite time.

We denote by $\mathcal{A}$ the set of admissible controls.

Definition 2.7.2. *For $a < b$, when there is triad of processes $X^{a,b} = \{X_t^{a,b}\}_{t\geq 0}$, $(U^{a,b}, D^{a,b}) = (\{U_t^{a,b}\}_{t\geq 0}, \{D_t^{a,b}\}_{t\geq 0})$ like the ones defined in 2.5.2 without finite explosion time, we say that $X^{a,b}, (U^{a,b}, D^{a,b})$ are an* admissible reflected process *(or just reflected process to keep it short) in $[a, b]$ and* admissible reflecting control *(or just reflecting control) respectively.*

We proceed to briefly describe the control problems that are treated in the thesis, without specifying most of the assumptions.

2.7.1 The ergodic problem

We are looking at an integral cost averaged over infinite time. The cost of the controls are *singular* and constant and there is a running integral continuous cost c.

Definition 2.7.3. *We define the ergodic value function as*

$$G(x) = \inf_{\eta\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T}\mathbf{E}_x \left(\int_0^T c(X_s^\eta)ds + q_u U_T + q_d D_T\right),$$

where $\eta = (U, D) \in \mathcal{A}$.

The objective is to find a pair $a^* \leq b^*$ such that $(U^{a^*,b^*}, D^{a^*,b^*})$ realizes the infimum over all $\eta \in \mathcal{A}$ and to give a computable expression of G. This problem is treated in Chapter 3 for Itô-diffusions and in Chapter 5 for Lévy processes.

2.7.2 The ϵ-discounted problem

In this case, there is a discount $\epsilon > 0$ representing depreciation of value over time. The ϵ-discounted value function G is defined as:

$$G(x) = \inf_{(U,D)\in\mathcal{A}} \mathbf{E}_x \left(\int_0^\infty e^{-\epsilon s} \left(c(X_s^{U,D})ds + q_u dU_s + q_d dD_s\right) + q_u u_0 + q_d d_0\right).$$

The objectives are the same as in the ergodic problem, plus studying the abelian limit. This is done in Chapter 5 for Lévy processes.

2.8 Relationship with Dynkin games

For Itô-diffusions, the approach to solve the ergodic singular control problem is quite straightforward as the ergodic limit depends on m' and S' (this is treated in Chapter 5). However this is not the case for general Lévy processes. To keep it short, the probabilistic properties of the reflecting control $(U^{a,b}, D^{a,b})$ in both cases depend on:

$$\mathbf{P}_x(X_{\gamma_{(a,b)^c}} \geq b),\ x \in \mathbb{R},$$

where $\gamma_{(a,b)^c}$ is the first exit from the set (a, b) which is not known for every Lévy process. The roadmap to solve singular control problems in this thesis for Lévy processes is:

- Define an auxiliary problem.
- Utilize its properties to prove that for the discounted singular control problem the value G_ϵ is reached in the set of reflecting controls and in some cases give the explicit values of the barriers of the optimal controls.
- Prove that the abelian limit holds to deduce that the value G is also reached in the set of reflecting controls.

The auxiliary problem is a *Dynkin game.*

Definition 2.8.1. *[Peskir(2009), Ekström and Peskir(2008)] Let $\mathbf{X} = \{\mathbf{X}_t\}_{t\geq 0}$ be a strong Markov process, and let G_1, G_2 and G_3 be continuous functions satisfying $G_1 \leq G_2 \leq G_3$. Consider a stopping game where the sup-player chooses a stopping time τ to maximize, and the inf-player chooses a stopping time σ to minimize, the expected payoff*

$$M_{\mathbf{x}}(\tau, \sigma) = \mathbf{E}_{\mathbf{x}}\left(G_1(\mathbf{X}_\tau)\mathbf{1}_{\{\tau<\sigma\}} + G_2(\mathbf{X}_\sigma)\mathbf{1}_{\{\tau=\sigma\}} + G_3(\mathbf{X}_\sigma)\mathbf{1}_{\{\tau>\sigma\}}\right).$$

The Dynkin game is the problem consisting in finding two stopping times (τ^*, σ^*) s.t.

$$M_{\mathbf{x}}(\tau, \sigma^*) \leq M_{\mathbf{x}}(\tau^*, \sigma^*) \leq M_{\mathbf{x}}(\tau^*, \sigma), \quad \textit{for all } \tau, \sigma \textit{ stopping times},$$

which is equivalent to

$$\sup_\tau \inf_\sigma M_{\mathbf{x}}(\tau, \sigma) = \inf_\sigma \sup_\tau M_{\mathbf{x}}(\tau, \sigma) = M_{\mathbf{x}}(\tau^*, \sigma^*).$$

In our case $\mathbf{X} = (X, I, Z)$ with $\{Z_t = r + t\}_{t\geq 0}$ and $\left\{I_t = w + \int_0^t e^{-\epsilon Z_s} c'(X_s)ds\right\}_{t\geq 0}$ and the functions $G_1, G_2 = G_3$ are defined as

$$G_1(x, w, r) = w - q_u e^{-\epsilon r},$$
$$G_2(x, w, r) = w + q_d e^{-\epsilon r}.$$

This Dynkin game is a particular case of [Stettner (1982)] , the properties used are based on [Peskir(2009)] and [Ekström and Peskir(2008)]. Moreover, the relationship between two-sided singular control problems and Dynkin games was first proved (under a different framework) in [Karatzas and Wang (2003)]. The relationship boils down to the fact that the nature of this Dynkin game, the functions τ^* and σ^* are the first entry to a negative, respectively positive, half line and their borders define a^* and b^*. An analysis of this relationship is given in 5.1.

2.9 Mean field games for two-sided singular control problems

We are also interested in giving our singular control problems a MFG component. The analysis of the selected literature is in chapters 4 and 6. As mentioned in the introduction, the MFG can be summarized as the limit when $N \to \infty$ of an N player game where the objective is the search of an equilibrium.

2.9.1 Itô-difusions

In this setting, we assume that the influence of the *infinite pool of players* affects the cost c now depending in two variables, in the second one. The setting is the following:

- In search of equilibrium, we assume that the pool of players considers a reflecting control $(U^{c,d}, D^{c,d})$.
- Now the cost function is not of the form, $c(X_t^{a,b})$ but $c(X_t^{a,b}, \mathbf{E}_x(f(X^{c,d})))$ (under reasonable hypotheses).
- We are in the search of a pair of points $a < b$ such that if a new player enters the market, his/her most rational action is to imitate the actual pool of players.
- In other words, we are interested in giving conditions for the existence and uniqueness of

a pair (a, b) such that

$$(U^{a,b}, D^{a,b}) \in \operatorname*{arg\,min}_{\eta=(U,D)\in\mathcal{A}} \left\{ \limsup_{T\to\infty} \frac{1}{T}\mathbf{E}_x \left(\int_0^T c\big(X_s^\eta, \mathbf{E}_x(f(X_s^{a,b}))\big)ds + q_u U_T + q_d D_T \right) \right\}.$$

- Finally, we define an adequate N-player game and study its convergence.

The study of the problem is analytic in nature, in other words, the equilibrium is characterized as the root of an equation.

2.9.2 Lévy processes

In this case, we do not assume that the pool of players considers a reflecting strategy. We study the MFGs associated to the discounted and ergodic case.

- We assume that the pool of players considers a control $(\tilde{U}, \tilde{D}) \in \mathcal{A}$ such that the controlled process $X^{\tilde{U},\tilde{D}}$ has compact support and converges in distribution to a stationary random variable $X_\infty^{\tilde{U},\tilde{D}}$.
- Now the cost function is not of the form, $c(X_t^{a,b})$ but $c(X_t^{a,b}, \mathbf{E}_x(f(X_\infty^{\tilde{U},\tilde{D}})))$ (under reasonable hypotheses). Clearly, now the value and discounted value functions also depend on one more variable.
- We are in search of a pair of points $a \leq b$ (the only case that $a = b$ is possible, is the ergodic one when the process has bounded variation) such that if a new player enters the market, his/her most rational action is to imitate the actual pool of players.
- That is, for the discounted case, we want a pair $a < b$ such that

$$\begin{aligned} &G_\epsilon(x, \mathbf{E}_x(f(X_\infty^{a,b})) \\ &\quad = \mathbf{E}_x \left(\int_0^\infty e^{-\epsilon s}(c(X_s^{U^{a,b},D^{a,b}}, \mathbf{E}_x(f(X_\infty^{a,b})))ds + q_u dU_s + q_d dD_s \right) + u_0^{a,b} q_u + d_0^{a,b} q_d, \end{aligned}$$

  and for the ergodic case, we want a pair $a \leq b$ such that

$$G(x, \mathbf{E}_x(f(X_\infty^{a,b})) = \limsup_{T\to\infty} \frac{1}{T}\mathbf{E}_x \left(\int_0^T (c(X_s^{U^{a,b},D^{a,b}}, \mathbf{E}_x(f(X_\infty^{a,b})))ds + q_u dU_s + q_d dD_s \right)$$

- We are also interested in the convergence (at least by taking a subsequence) in the abelian sense of discounted equilibrium control to an ergodic equilibrium.

We give reasonable hypotheses to assure the existence of equilibrium points and characterize them as an integro-differential equation. The main tool is to use the Brouwer fixed point Theorem in the adjoint Dynkin game and prove that the ϵ-discounted equilibrium points converge (or at least have a convergent subsequence) to the ergodic equilibrium.

Chapter 3

Two-sided ergodic control for Itô-diffusions

Abstract

We treat the optimization problem mentioned in 2.7.1 for an Itô-diffusion. More precisely, we study the problem [Alvarez(2018)], use a HJB equation as verification theorem and describe the solutions of the problem as the unique pair of roots of a non-linear system.

We treat the problem posed in 2.7.1. That is, an ergodic two-sided control problem with an underlying stochastic process $X = \{X_t\}_{t\geq 0}$. In this chapter $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ is a filtered probability space that satisfy the usual assumptions. The process X is an Itô-diffusion satisfying the SDE:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \; X_0 = x_0. \tag{3.1}$$

We refer to the more technical details in Section 3.2.

This chapter is organized as follows. In Section 3.1, we give some historical remarks and motivations of the two-sided ergodic control problem. In Section 3.2, we postulate the problem [Alvarez(2018)] and make a resume of the first part of the article. In Section 3.4, we use [Christensen et al. (2023)] to extend the results [Alvarez(2018)] to a more general class of controls. Finally, Section 3.5 has examples when the underlying process is a Ornstein–Uhlenbeck process and a Brownian motion with drift.

3.1 Introduction

In the two-sided ergodic control problem, we are looking at an integral cost averaged over infinite time, thus rendering unimportant any initial finite action. These problems have applications when sustainability is on play. To be more specific, with respect to applications of singular control results, we mention studies focusing on cash flow management that investigate optimal dividend distribution, recapitalization, or a combination of both, while considering risk neutrality. See, for example, [Asmussen and Taksar (1997)], [Højgaard and Taksar (2001)], [Jeanblanc-Picqué and Shiryaev (1995)], [Paulsen (2008)], [Peura and Keppo (2006)], and [Shreve et al. (1984)]. Moreover, the ergodic model comes pretty handy in problems of finding sustainable harvesting policies (see [Clark (2010), Chapter I]). As examples where a solution of a free boundary problem defined by a HJB is used to solve ergodic control problems, we refer to:

- [Karatzas (1983)] This is the first time that the two-sided ergodic control problem 2.7.1 is posed. As an application, the author proposes that the controlled process can be thought as a demand which has to be met or at least be close to a certain quantity. The underlying process is a Brownian motion. Due to the nature of its infinitesimal generator the problem is directly solved by studying the free boundary problem defined by the HJB equation.
- [Menaldi and Robin (1984)] In this article, the authors study an one-sided ergodic control problem. Although it differs from the problems of study in this thesis, we remark that in this problem the underlying process is an one-dimensional diffusion satisfying (3.2) (under the appropriate hypothesis). Again, due to the nature of the infinitesimal generator, the ergodic problem is solved quite directly.
- [Weerasinghe (2002)] In this article, the author treats a two-sided problem for a specific family of continuous diffusions. The main structural differences with [Alvarez(2018)] are that the family of processes that they study follow the SDE :

  $$dX_t^{U,D} = \mu(t)dt + \sigma(t)dW_t + dU_t - dD_t, \quad X_0 = x_0, \qquad U, D \text{ increasing adapted processes}$$

  and the optimization is not only on the processes U, D but also on the drift and volatility (obviously under a restricted family of functios). Moreover, the cost function is under different assumptions. The procedure is again to formulate a verification theorem and by analytic arguments show that there is a candidate.
- [Jack and Zervos (2006)] In this article the authors study a two-sided problem with an underlying diffusion. It is remarkable that the costs of the controls, although bounded by a constant, are not constant. Due to its more general approach, the assumptions given for the diffusion and the cost are more restrictive and the the existence of optimal controls is not expressed explicitly.
- [Weerasinghe (2007)] In this article the main tool to solve the ergodic two-sided control problem is to solve a discounted problem similar to 2.7.2 and use the Abelian limit. We

also remark that the author postulates a restrained optimization problem in the sense that there is a constant m such that the controls must satisfy

$$\limsup_{T\to\infty} \frac{1}{T}\mathbf{E}(U_t + D_t) \leq m,$$

and shows that there is a m^* for the restriction such that the ergodic problem is equivalent to the ergodic restrained problem. Furthermore the author shows that depending on the assumptions, it might be optimal to not control the process.

- [Menaldi and Robin (2013)] In this work, the underlying process is a multidimensional Gaussian Process added to a Compound Poisson process. The authors use the abelian limit to show that there exists an optimal control. The proofs and further extensions were left for future works.
- [Wu and Chen (2017)] In this work the authors study a n-dimensional Brownian motion. Due to the nature of the infinitesimal generator of the Brownian motion and the symmetric assumptions on the cost functional, the study of the problem is radial.
- [Arapostathis et al. (2019)] In this article, the controls are a set of parameters on a compact set that affect the drift. Although our techniques and framework greatly differs from the problems studied in this thesis we wanted to mention this paper because the underlying process is a jump-diffusion. The main objectives are to prove the existence and in some cases the uniqueness of the solution.
- [Kunwai et al. (2022)] The authors work here with a diffusion satisfying the SDE (3.2) (under certain hypothesis for μ and σ). The problem proposed is similar to the one in this chapter, the structural difference lies in the fact that one of the controls has a "negative cost" (a reward).

3.2 Framework for the Ergodic Control problem

In this Section we give the framework for our ergodic two-sided control problem which is based on [Alvarez(2018)] and [Christensen et al. (2023)].

3.2.1 Itô-diffusion

Let us consider a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ that satisfies the usual assumptions. In order to define the underlying diffusion, consider the functions $\mu\colon \mathbb{R} \to \mathbb{R}$ and $\sigma\colon \mathbb{R} \to (0,\infty)$ assumed to be locally Lipschitz. Under these conditions, as mentioned in 2.2.3, the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,\ \ X_0 = x \tag{3.2}$$

has a unique strong solution up to an explosion time, that we denote $X = \{X_t \colon t \geq 0\}$ (see [Protter (2005), Theorem V.38]). Observe that our framework includes quadratic coefficients. Alternatively, our results can be formulated in the framework of weak solutions as in [Alvarez(2018)]. As usual, we define the infinitesimal generator of the process X as

$$\mathcal{L}_X = \frac{1}{2}\sigma^2(x)\frac{d^2}{d^2x} + \mu(x)\frac{d}{dx}.$$

For more details, see Section 2.4. As in Definition 2.2.10, we denote the density of the scale function $S(x)$ w.r.t the Lebesgue measure as

$$S'(x) = \exp\left(-\int^x \frac{2\mu(u)}{\sigma^2(u)}du\right),$$

and the density of the speed measure $m(x)$ w.r.t the Lebesgue measure as

$$m'(x) = \frac{2}{\sigma^2(x)S'(x)}.$$

As mentioned above, the underlying process is controlled by a pair of processes, the admissible controls, that drive it to a convenient region. As in Definition 2.7.1, we define *admissible control* and use the same notations for $\eta = (U = \{U_t\}_{t\geq 0}, D = \{D_t\}_{t\geq 0})$, $\{X^\eta\}$ and the set $\mathcal{A}$. Nevertheless we rewrite its definition for this particular calss of processes.

Definition 3.2.1. *An* admissible control *is a pair of non-negative* $\mathbf{F}$*-adapted processes* $\eta = (U = \{U_t\}_{t\geq 0}, D = \{D_t\}_{t\geq 0})$ *such that:*

(i) *Each process* U, D *is right-continuous and non decreasing almost surely.*

(ii) *For each* $t \geq 0$ *the random variables* U_t *and* D_t *have finite expectation.*

(iii) *For every* $x \in \mathbb{R}$ *the stochastic differential equation*

$$dX_t^\eta := \mu(X_t^\eta)dt + \sigma(X_t^\eta)dW_t + dU_t - dD_t, \quad X_0 = x \tag{3.3}$$

has a unique strong solution with no explosion in finite time.

We denote by $\mathcal{A}$ *the set of admissible control.*

Note that condition (iii) is satisfied, for instance, when the coefficients are globally Lipschitz. (See the remark after Theorem V.38 in [Protter (2005)]). Observe also that condition (ii) is not a real restriction, as, for instance, the integral in the cost function $G(x)$ in (3.6) that we aim to minimize, in case of having infinite expectations, is infinite.

A relevant sub-class of admissible controls is the set of reflecting controls. For the sake of clarity, we rewrite the definition of reflecting controls given in 2.5.3.

Definition 3.2.2. *For $a < b$ denote by $X^{a,b} = \{X_t^{a,b} \colon t \geq 0\}$ the strong solution of the stochastic differential equation with reflecting boundaries at a and b:*

$$dX_t^{a,b} = \mu(X_t^{a,b})dt + \sigma(X_t^{a,b})dW_t + dU_t^{a,b} - dD_t^{a,b}, \qquad X_0 = x.$$

Here $U^{a,b} = \{U_t^{a,b}\}, D^{a,b} = \{D_t^{a,b}\}$, are the local times of the reflected diffusion in the interval $[a, b]$. They are continuous non-decreasing processes that increase, respectively, only when the solution visits a or b and make the controlled diffusion satisfy the condition $a \leq X_t^{a,b} \leq b$, a.s. for all $t \geq 0$. As the above equation has a strong solution (see [Lions and Sznitman (1984), Theorem 3.1] or [Saisho (1987), Theorem 5.1]), the pair $(U^{a,b}, D^{a,b}), (a, b)$ belongs to $\mathcal{A}$, $\mathbb{R}^2$ respectively, we call them reflecting controls *and* reflecting barriers *respectively. If $x \notin (a, b)$, we begin the policy by sending the process to the closest point of the interval $[a, b]$ at time $t = 0$.*

3.2.2 Cost function

We introduce below the cost function $c(x)$ to be considered in the optimization problem.

Assumption 3.2.1. *Assume that $c \colon \mathbb{R} \to \mathbb{R}_+$ is a continuous function, and the positive constants q_u, q_d are the unit cost of using the associated controls. Assume that, there exist a value x_m such that*

$$c(x) \geq c(x_m) \geq 0, \qquad \textit{for all } x \in \mathbb{R},$$

and constants $K \geq 0$ and $\alpha > 0$ such that

$$c(x) + K \geq \alpha|x|, \qquad \textit{for all } x \in \mathbb{R}. \tag{3.4}$$

Consider the maps

$$\pi_1(x) = c(x) + q_d\mu(x), \qquad \pi_2(x) = c(x) - q_u\mu(x),$$

and assume:

(i) *There exists a unique real number $x_i^0 = \arg\min\{\pi_i(x) \colon x \in \mathbb{R}\}$ so that $\pi_i(\cdot)$ is decreasing on $(-\infty, x_i^0)$ and increasing on (x_i^0, ∞), where $i = 1, 2$.*

(ii) *The following limits hold:*

$$\lim_{x\to\infty} \pi_1(x) = \lim_{x\to-\infty} \pi_2(x) = \infty. \tag{3.5}$$

Definition 3.2.3. *We define the ergodic value function as*

$$G(x) = \inf_{\eta\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T}\mathbf{E}_x \left(\int_0^T c(X_s^\eta)ds + q_u U_T + q_d D_T \right), \tag{3.6}$$

where $\eta = (U, D)$ is an admissible control in $\mathcal{A}$.

3.3 Optimality within reflecting controls

The existence of the unique pair of optimal controls within the class of reflecting controls was obtained by [Alvarez(2018)], we extract the result (with some changes in the notations but most importantly we add the condition (3.4)) and its proof from the article as it illustrates how to find optimal controls in similar frameworks when the underlying process is an Itô diffusion. It is noticeable, due to the nature of the underlying processes in this chapter, that the ergodic limit has an explicit expression. We remark that all of this section is extracted from the aforementioned paper.

3.3.1 Conditions for optimal barriers

To proceed, first, the author studied the ergodic limit within the class of reflecting controls and secondly he made an analytic study of the obtained expression to find its minimum.

Lemma 3.3.1. *Under Assumption 3.2.1, If $a < b$ then:*

$$\lim_{T\to\infty} \frac{1}{T}\mathbf{E}_x \left(\int_0^T c(X_s^{a,b})ds + q_u U_T^{a,b} + q_d D_T^{a,b} \right)$$
$$= \frac{1}{m(a,b)} \left[\int_a^b c(u)m(du) + \frac{q_u}{S'(a)} + \frac{q_d}{S'(b)} \right] =: C(a,b).$$

Proof. Let $f \in C^2(\mathbb{R})$ be an arbitrary twice continuously differentiable function and let $x \in [a, b]$. Applying Doléans-Dade-Meyer change of variable formula (or Itô formula and using the linearity in the stochastic integrands) to the function f we obtain:

$$f(X_T^{a,b}) = f(x) + \int_0^T (\mathcal{L}f)(X_s^{a,b})ds + \int_0^T \sigma(X_s^{a,b})dW_s + f'(a)U_T^{a,b} - f'(b)D_T^{a,b}.$$

Since $X_t^{a,b}$ is continuous in $[a,b]$ and the processes $U_t^{a,b}$, $D_t^{a,b}$ increase only at the boundaries a and b respectively, by taking expectation and reordering the terms, we get

$$\mathbf{E}_x(f(X_T^{a,b}) - f(x)) = \mathbf{E}_x\left(\int_a^b (\mathcal{L}f)(X_s^{a,b})ds + f'(a)U_T^{a,b} - f'(b)D_T^{a,b}\right). \tag{3.7}$$

Dividing by T, taking the limit $T \to \infty$ and using Theorem 2.5.3 we get

$$0 = \int_a^b (\mathcal{L}f)(t)\frac{m'(t)}{m(a,b)}ds + \lim_{T\to\infty} \mathbf{E}\left(f'(a)U_T^{a,b} - f'(b)D_T^{a,b}\right).$$

On the other hand, from we definition of $S'(x)$ we have

$$\left(\frac{f'(t)}{S'(t)}\right)' = \frac{f''(t)}{S'(t)} + \frac{2\mu(t)f'(t)}{\sigma^2(t)} = (\mathcal{L}f)(t)m'(t).$$

Thus equation (3.7) can be rewritten as:

$$0 = \frac{f'(b)}{S'(b)} - \frac{f'(a)}{S'(a)} + \lim_{T\to\infty} \mathbf{E}\left(f'(a)U_T^{a,b} - f'(b)D_T^{a,b}\right).$$

Due to the fact f is an arbitrary twice continuously differentiable function we deduce

$$\frac{1}{S'(a)} = \lim_{T\to\infty} \mathbf{E}\left(U_T^{a,b}\right), \qquad \frac{1}{S'(b)} = \lim_{T\to\infty} \mathbf{E}\left(D_T^{a,b}\right). \tag{3.8}$$

Finally, applying Theorem 2.5.3 and equation (3.8) we conclude the Lemma. □

We proceed to characterize the optimal reflecting control, by differentiation and using the equality

$$\int_a^b \mu(t)m'(t)dt = \frac{1}{S'(a)} - \frac{1}{S'(b)}, \tag{3.9}$$

we deduce the next Corollary.

Corollary 3.3.2. *If a pair $a^* < b^*$ of boundaries minimizing the expected long-run average cumulative costs exist, it has to satisfy the ordinary first order conditions*

$$C(a^*, b^*) = \pi_2(a^*) = \pi_1(b^*), \tag{3.10}$$

which can be re-expressed as

(i) $I_1(a^*, b^*) := \displaystyle\int_{a^*}^{b^*} (\pi_1(t) - \pi_1(b^*))\, m'(t)dt + \frac{q_u + q_d}{S'(a^*)} = 0,$

(ii) $I_2(a^*, b^*) := \int_{a^*}^{b^*} (\pi_2(t) - \pi_2(a^*))\, m'(t)dt + \frac{q_u + q_d}{S'(b^*)} = 0.$

Theorem 3.3.3. *The optimality conditions of Corollary 3.3.2 have a uniquely determined solution* $(a^*, b^*) \in (-\infty, x_2^0) \times (x_1^0, \infty)$.

Proof. First of all, due to 3.2.1 (i), notice that the items (i) and (ii) of Corollary 3.3.2 are not zero if $b^* \leq x_1^0, \quad a^* \geq x_2^0$ respectively. Second, by observing

$$I_1(a, b) - I_2(a, b) = m(a, b)(\pi_2(a) - \pi_1(b)),$$

we deduce that proving (i) and (ii) is equivalent to proving (i) and the property

$$\pi_2(a^*) = \pi_1(b^*). \tag{3.11}$$

On the other hand, due to the fact π_2, π_1 are monotone in $(-\infty, x_2^0], [x_1^0, \infty)$ respectively and unbounded above if we restrict their domains to these half-lines, we deduce that there is a half line $(-\infty, \hat{a}] \subset (-\infty, x_2^0]$ such that for every $a \leq \hat{a}$ there is an unique b_a such that $\pi_1(b_a) = \pi_2(a)$. In order to establish the sets where b_a is well-defined, consider first the case where $x_1^0 \geq x_2^0$. If $\pi_1(x_1^0) \geq \pi_2(x_2^0)$ then our assumptions guarantee that there is a unique threshold $\hat{a} = \{x \in (-\infty, x_2^0]\}$ so that b_a is well-defined for all $a \leq \hat{a}$. Analogously, if $\pi_1(x_1^0) \leq \pi_2(x_2^0)$, then the function b_a is well-defined for all $a \leq x_2^0$. Consider now instead the case where $x_1^0 \leq x_2^0$. It is then clear that our assumptions guarantee that if $\pi(x_1^0) \geq \pi_2(x_1^0)$ then we again oserve that b_a is well-defined for all $a \leq \hat{a}$. If $\pi_2(x_2^0) \geq \pi_1(x_1^0)$, then b_a is well-defined for all $a \leq x_2^0$. Finally, if either $\pi_2(x_1^0) \geq \pi_1(x_1^0) \geq \pi_2(x_2^0)$ or $\pi_1(x_2^0) \geq \pi_2(x_2^0) \geq \pi_1(x_1^0)$, then there is a unique intersection point $\hat{x} \in [x_1^0, x_2^0]$ at which $\pi_1(\hat{x}) = \pi_2(\hat{x})$ and b_a is well-defined for all $a \leq \hat{x}$ and satisfies the condition $b_{\hat{x}} = \hat{x}$. In fact, due to the continuity and monotonocity of the functions the map

$$a \to b_a, \quad a \in (-\infty, \hat{a}], \tag{3.12}$$

is continuous. Then, by equation (3.11) we deduce it is enough to prove that the function

$$g : (-\infty, \hat{a}] \to \mathbb{R}, \quad g(a) = \int_a^{b_a} (\pi_1(t) - \pi_1(b_a))m'(t)dt + \frac{q_d + q_u}{S'(a)}, \tag{3.13}$$

has an unique root. Observe, due to equation (3.9), the function g is can be rewritten as:

$$g(a) = \int_a^{b_a} \pi_1(t)m'(t)dt - \pi_1(b_a)m(a, b) + \frac{q_d}{S'(b_a)} + \frac{q_u}{S'(a)}.$$

We now plan to prove that $g(a) > 0$ at the upper boundary where b_a is defined. Consider first the case where $\pi_1(x_1^0) \geq \pi_2(x_2^0)$. Utilizing (3.11) shows

$$\begin{aligned}
g(a) &= \int_{\hat{a}}^{x_1^0} c(t)m'(t)dt - \pi_1(x_1^0)m(\hat{a}, x_1^0) + \frac{q_d}{S'(x_1^0)} + \frac{q_u}{S'(\hat{a})} \\
&= \int_{\hat{a}}^{x_1^0} \pi_1(t)m'(t)dt - \pi_1(x_1^0)m(\hat{a}, x_1^0) + \frac{q_d + q_u}{S'(\hat{a})},
\end{aligned}$$

since $x_1^0 = \arg\min(\pi_1(x))$. Consider now the case either $x_1^0 \geq x_2^0$ and $\pi_1(x_1^0) \leq \pi_2(x_2^0)$ or $x_1^0 \leq x_2^0$ and $\pi_1(x_2^0) \leq \pi_2(x_2^0)$. In those cases we find that

$$\begin{aligned}
g(x_2^0) &= \int_{x_2^0}^{b_{x_2^0}} c(t)m'(t)dt - \pi_1(b_{x_2^0})m(x_2^0, b_{x_2^0}) + \frac{q_d}{S'(b_{x_2^0})} + \frac{q_u}{S'(x_2^0)} \\
&= \int_{x_2^0}^{b_{x_2^0}} \pi_2(t)m'(t)dt - \pi_2(x_2^0)m(x_2^0, b_{x_2^0}) + \frac{q_d + q_u}{S'(b_{x_2^0})} > 0,
\end{aligned}$$

since $x_2^0 = \arg\min(\pi_2(x))$. Finally if $x_1^0 \leq x_2^0$ and either $\pi_2(x_1^0) \geq \pi_1(x_1^0) \geq \pi_2(x_2^0)$ or $\pi_1(x_2^0) \geq \pi_2(x_2^0) \geq \pi_1(x_1^0)$ holds, then

$$g(\hat{x}) = \frac{q_d + q_u}{S'(\hat{x})} > 0,$$

proving the alleged positivity of $g(a)$ at the upper boundary of the set where b_a is defined. We now plan to establish that equation $g(a) = 0$ has a unique root on $(-\infty, x_2^0]$ by establishing that $g(a)$ is monotonically increasing on its domain and tends to $-\infty$ as $a \to -\infty$. To this end, assume that $a_1 < a_2$ and, therefore, that $b_1 > b_2$, where $b_i := b_{a_i},\ i = 1, 2$. Utilizing the definition of the function g, as well as the identities (3.11) and $\pi_2(a_i) = \pi_1(b_i), i = 1, 2$ yields

$$\begin{aligned}
g(a_2) - g(a_1) = &\int_{a_1}^{a_2} (\pi_2(a_1) - \pi_2(t))m'(t)dt + \int_{b_2}^{b_1} (\pi_1(b_1) - \pi_1(t))m'(t)dt \\
&+ (\pi_2(a_1) - \pi_2(a_2))m(a_2, b_2) > 0,
\end{aligned}$$

demonstrating that $g(a)$ is monotonically increasing. Moreover, since

$$g(a_2) - g(a_1) > (\pi_2(a_1) - \pi_2(a_2))m(a_2, b_2) \to \infty$$

as $a_1 \to -\infty$ we notice that $\lim_{a_1 \to -\infty} g(a_1) = -\infty$. Combining this observation with the monotonicity and continuity of g and the positivity of g at the upper boundary of its domain demonstrates that the equation $g(a) = 0$ has a unique root $a^* \in (-\infty, x_2^0]$. Moreover, since $g(a^*) = I_1(a^*, b^*) = I_2(a^*, b^*) = 0$, where $b^* = b_{a^*}$, we find that the pair (a^*, b^*) constitutes the unique root of the optimality conditions (i) and (ii) of Corollary 3.3.2. □

3.3.2 Adjoint free boundary problem, existence and uniqueness of optimal reflecting controls

As seen in the introduction of this chapter, in the search of optimal controls, a free boundary problem arises naturally in the form of a HJB equation. Our objective is now to determine the twice continuously differentiable function $u : \mathbb{R} \to \mathbb{R}_+$ as well as the two boundaries $a < 0 < b$ and the parameter $\lambda > 0$ solving the free boundary problem:

$$\begin{aligned}
&(\mathcal{L}u)(x) + c(x) = \lambda, \quad x \in (a, b),\\
&u(x) = q_d(x - b) + u(b), \quad x \geq b,\\
&u(x) = q_u(a - x) + u(a), \quad x \leq a.
\end{aligned} \tag{3.14}$$

We find by integrating over the interval (a, b) and invoking the boundary conditions $u'(a) = -q_u,\ v'(b) = q_d$ that

$$\lambda m(a, b) - \int_a^b c(x)m'(x)dx = \int_a^b (\mathcal{L}u)(x)m'(x)dx = \frac{q_d}{S'(b)} + \frac{q_u}{S'(a)}.$$

Consequently

$$\lambda = \frac{1}{m(a, b)} \left(\int_a^b c(x)m'(x)dx + \frac{q_d}{S'(b)} + \frac{q_u}{S'(a)} \right). \tag{3.15}$$

Finally, imposing the conditions $v''(a) = v''(b) = 0$ guaranteeing the second order differentiability of the value across the boundaries implies that

$$\lambda = c(b) + q_d\mu(b) = c(a) - q_u\mu(a). \tag{3.16}$$

We observe these conditions coincide with the ones in equation 3.11. Moreover for the unique values (a^*, b^*) defined in Theorem 3.3.3 we define

$$\lambda^* := C(a^*, b^*) = \pi_1(b^*) = \pi_2(a^*). \tag{3.17}$$

We can study more properties of the function u defined in the free boundary problem that will be useful in the next Section to prove that the class of reflecting controls reach a global minimum in a wider class of controls.

Proposition 3.3.4. *For the pair (a^*, b^*) defined in Theorem 3.3.3 and λ^* in (3.17) the adjoint function u defined in the free boundary problem satisfies $-q_u \leq u'(x) \leq q_d$ for all $x \in \mathbb{R}$.*

Proof. It is clear that is enough to prove that $-q_u \leq u'(x) \leq q_d$ on (a^*, b^*). First notice

$$(((\mathcal{L}u)(x) - q_d\mu(x))m'(x))' = \frac{v'(x) - q_d}{S'(x)}.$$

Second, we obtain by invoking the boundary condition $u'(b^*) = q_d$, and $\lambda^* = \pi_1(b^*)$ that

$$\frac{u'(x) - q_d}{S'(x)} = \int_x^{b^*} (\pi_1(t) - \pi_1(b^*))m'(t)dt.$$

We first prove that the integral in this expression is non-positive on $[a^*, b^*]$. To see that this is indeed true, we first notice that if $\lim_{x\to-\infty} \pi_1(x) \leq \pi_1(b^*)$, then the integrand is always non-positive proving the statement in that case. If, however, $\lim_{x\to-\infty} \pi_1(x) > \pi_1(b^*)$, then our assumptions on the function π_1 guarantee that there exists a uniquely defined state $y_1 = \{x < x_1^0 : \pi_1(x) = \pi_1(b^*)\}$. However, since the integrand is non-positive for all $y \in [y_1, b^*]$ and the integral is non-increasing for $a^* \leq x \leq \max(a^*, y_1)$ and (i) of Corollary 3.3.2, we notice that the integral is non-positive in that case as well. In order to complete the proof it is sufficient to show that

$$\frac{u'(x) - q_d}{S'(x)} \geq -\frac{q_u + q_d}{S'(x)} \tag{3.18}$$

for all $x \in [a^*, b^*]$. To this end consider now the function

$$D(x) = \int_x^{b^*} (\pi_1(t) - \pi_1(b^*))m'(t)dt + \frac{q_u + q_d}{S'(x)}.$$

It is clear that $D(a^*) = 0$, $D(b^*) = \frac{q_u+q_d}{S'(b^*)} > 0$, and $D'(x) = (\pi_1(b^*) - \pi_2(x))m'(x) = (\pi_2(a^*) - \pi_2(x))m'(x)$. Two cases arise. If $\lim_{x\to\infty} \pi_2(x) \leq \pi_2(a^*)$, then $D'(x) \geq 0$ for all $x \in [a^*, b^*]$ and we are done. If, however, $\lim_{x\to\infty} \pi_2(x) > \pi_2(a^*)$, then $D'(x) \geq 0$ for all $x \in [a^*, y_2]$ and $D'(x) < 0$ for all $x > y_2$ where $y_2 = \{x > x_2^0 : \pi_2(x) = \pi_2(a^*)\}$. Consequently, we notice that $D(x) \geq 0$ for all $x \in [a^*, b^*]$ in that case as well, completing the proof of our theorem. □

Finally, we are ready to prove that the conditions in Theorem 3.3.3 gives a unique optimal reflecting policy and the infimum of the function $C(a, b)$ does not depends on the initial value of the process X and is equal to λ^* defined in (3.17).

Theorem 3.3.5. *The pair $(a^*, b^*) \in (-\infty, x_2^0) \times (x_1^0, \infty)$ is a global minimum on $C(a, b)$ and $C(a, b) \geq C(a^*, b^*)$ for all $-\infty < a < b < \infty$. Consequently, $\lambda \geq \lambda^*$ for all $-\infty < a < b < \infty$ and $\eta^{a^*,b^*} := (U^{a^*,b^*}, D^{a^*,b^*})$ constitutes an optimal singular control within the considered class of reflection controls*

Proof. Let us first investigate the behavior of the function $I_1(a,b)$. If $b_2 > b_1 > a$, then

$$I_1(a,b_1) - I_1(a,b_2) = (\pi_1(b_2) - \pi_1(b_1))m(a,b_1) + \int_{b_1}^{b_2} (\pi_1(b_2) - \pi_1(t))m'(t)dt.$$

It is now clear form our assumptions on π_1 that $I_1(a,b)$ is increasing on $(-\infty, x_1^0)$ and decreasing on (x_1^0, ∞) as a function of b. Moreover, if $b_2 > b_1 \geq x_1^0$, then $I_1(a,b_1) - I_2(a,b_2) > (\pi_1(b_2) - \pi_1(b_1))m(a,b_1) \to \infty$ as $b_2 \to \infty$. Consequently, $\lim_{b\to\infty} I_1(a,b) = -\infty$ for all $a \in \mathbb{R}$. Combining these observation with the identity $I_1(a,a) = \frac{q_u+q_d}{S'(a)} > 0$ implies that $I_1(a,b) = 0$ has a unique root $\hat{b}_a$ for any $a \in \mathbb{R}$ and $I_1(a,b)$ has the same sign as the function $(\hat{b}_a - b)$. Establishing now that $I_2(a,b)$ is increasing on $(-\infty, x_2^0)$, decreasing on (x_2^0, ∞) as a function of a and satisfies $\lim_{a\to\infty} I_2(a,b) = -\infty$ for all $b \in \mathbb{R}$ is completely analogous. Consequently, we notice that $I_2(a,b) = 0$ has a unique root $\hat{a}_b$ for any $b \in \mathbb{R}$ and $I_2(a,b)$ has the same sign as the function $(a - \hat{a}_b)$. Combining these observations with the uniqueness of the pair (a^*, b^*) and Corollary 3.3.2 completes the proof of the first claim of our Theorem. The second statement then follow directly from Lemma 3.3.1. □

Remark 3.3.1. *We have proved the existence and uniqueness of an optimal control in the set of reflecting controls defined as a root of a non-linear system of two equations (Theorems 3.3.3 and 3.3.5). Moreover we have a relationship between a free boundary problem and infimum value between the reflecting policies (see equation* (3.15)*). These are the results of the paper [Alvarez(2018)] that we will use but we remark that in the article the author also studied how increasing the volatility expands the interval* (a^*, b^*) *when the process has no drift.*

What is left to do is to prove that the optimal reflecting controls are in fact optimal within the class $\mathcal{A}$ defined in 3.2.1, we do this in the next Section, using the classical approach of postulating a verification theorem with the free boundary problem.

3.4 Optimality within admissible controls

In this Section we work with a wider class of controls, that is, we work with the set $\mathcal{A}$ defined in Definition 3.2.1. Optimality within the class $\mathcal{A}$ of càdlàg controls requires further analysis. As expected, and mentioned in [Alvarez(2018)], the optimal controls within class $\mathcal{A}$ are the same controls found in the class of reflecting controls. More precisely, it is clear that

$$\inf_{a<b} \lim_{T\to\infty} \frac{1}{T} \mathbf{E}_x \left(\int_0^T c(X_s^{a,b})ds + q_u U_T^b + q_d D_T^a \right) \geq G(x).$$

Then, to establish the optimality within $\mathcal{A}$ it is necessary to obtain the other inequality.

Theorem 3.4.1 (Verification)**.** *Consider a diffusion defined by* (3.2) *and a cost function* c *satisfying Assumption 3.2.1. Suppose that there exist a constant* $\lambda \geq 0$ *and a function* $u \in C^2(\mathbb{R})$ *such that*

$$(\mathcal{L}_X u)(x) + c(x) \geq \lambda, \qquad -q_u \leq u'(x) \leq q_d, \quad \text{for all } x \in \mathbb{R}. \tag{3.19}$$

Define the subset of admissible controls

$$\mathcal{B} = \left\{ \eta \in \mathcal{A} \colon \liminf_{T\to\infty} \frac{1}{T} \left| \mathbf{E}_x(u(X_T^\eta)) \right| = 0 \right\}. \tag{3.20}$$

Then,

$$\inf_{\eta\in\mathcal{B}} \limsup_{T\to\infty} \frac{1}{T} \mathbf{E}_x \left(\int_0^T c(X_s^\eta) ds + q_u U_T + q_d D_T \right) \geq \lambda. \tag{3.21}$$

Remark 3.4.1. *The consideration of the subclass* $\mathcal{B}$ *is not a restriction, as will be seen below. More precisely, it will be proved (using condition* (3.4)*), that controls in* $\mathcal{A} \setminus \mathcal{B}$ *give infinite values.*

Proof. Fix $T > 0$. For each $n \geq 1$ define the stopping times

$$T_n = \inf\{t \geq 0 \colon |X_t^\eta| \geq n\} \wedge T \nearrow T \quad a.s.$$

Using Itô formula for processes with jumps (observe that the diffusion X is continuous but the controls can have jumps, and in consequence the controlled processes X^η can have jumps),

$$\begin{aligned} u(X^\eta(T_n)) = u(x) &+ \int_0^{T_n} u'(X_{s-}^\eta) dX_s^\eta + \frac{1}{2} \int_0^{T_n} u''(X_{s-}^\eta) d\langle (X^\eta)^c, (X^\eta)^c \rangle_s \\ &+ \sum_{s\leq T_n} \left(u(X_s^\eta) - u(X_{s-}^\eta) - u'(X_{s-}^\eta) \triangle X_s^\eta \right). \end{aligned} \tag{3.22}$$

The r.h.s in (3.22) can be rewritten as

$$\begin{aligned} u(x) &+ \int_0^{T_n} (\mathcal{L}_X u)(X_{s-}^\eta) ds - \int_0^{T_n} \mu(X_{s-}^\eta) u'(X_{s-}^\eta) ds \\ &+ \int_0^{T_n} u'(X_{s-}^\eta) dX_s^\eta + \sum_{s\leq T_n} \left(u(X_s^\eta) - u(X_{s-}^\eta) - u'(X_{s-}^\eta) \triangle X_s^\eta \right). \end{aligned} \tag{3.23}$$

Using the fact that $u'(X_{s-}^\eta) = u'(X_s^\eta)$ in a set of total Lebesgue measure in $[0, T]$ almost surely, and that $\triangle X^\eta_s = \triangle U_s - \triangle D_s$, we rewrite (3.23) as

$$\begin{aligned} u(x) &+ \int_0^{T_n} (\mathcal{L}_X u)(X^\eta_{s-})ds + \int_0^{T_n} u'(X^\eta_{s-})\sigma(X^\eta_{s-})dW_s \\ &+ \int_0^{T_n} u'(X^\eta_{s-})d(U_s - D_s) \\ &+ \sum_{s \le T_n} \left(u(X^\eta_s) - u(X^\eta_{s-}) - u'(X^\eta_{s-})(\triangle U_s - \triangle D_s)\right). \end{aligned} \tag{3.24}$$

Therefore, denoting by U^c_s and D^c_s the continuous parts of the processes U_s and D_s respectively, and using the inequalities (3.19) in the hypothesis, we obtain

$$\begin{aligned} u(X^\eta(T_n)) &\ge u(x) + \lambda T_n - \int_0^{T_n} c(X^\eta_{s-})ds + \int_0^{T_n} u'(X^\eta_{s-})\sigma(X^\eta_{s-})dW_s \\ &\quad - \int_0^{T_n} q_u dU^c_s - \int_0^{T_n} q_d dD^c_s - \sum_{0 \le s \le T_n} (\triangle U_s q_u + \triangle D_s q_d) \\ &= u(x) + \lambda T_n - \int_0^{T_n} c(X^\eta_{s-})ds + \int_0^{T_n} u'(X^\eta_{s-})\sigma(X^\eta_{s-})dW_s \\ &\quad - q_u U_{T_n} - q_d D_{T_n}. \end{aligned}$$

Rearranging the terms above and taking the expectation we obtain

$$\mathbf{E}_x(u(X^\eta(T_n))) - u(x) + \mathbf{E}_x\left(\int_0^{T_n} c(X^\eta_{s-})ds + q_u U_{T_n} + q_d D_{T_n}\right) \ge \lambda \mathbf{E}_x(T_n).$$

Taking first limit as n tends to infinity, dividing then by T, and finally taking $\liminf$ as T goes to infinity we obtain (3.21) concluding the proof of the verification theorem. □

We now use the free boundary problem defined in (3.14) to conclude this section

Theorem 3.4.2. *Consider a diffusion defined by* (3.2) *and a cost function c satisfying Assumption 3.2.1. Then, the reflecting controls with levels given in in Theorem 3.3.5 minimize the ergodic value G in* (3.6) *within the set $\mathcal{A}$ of admissible controls.*

Proof. Take u as the solution of the free boundary problem (3.14). In view of Theorem 3.4.1, we need to prove that the infimum of the ergodic value defining $G(x)$ is realized in the set $\mathcal{B}$. Take then $\eta \in \mathcal{A} \setminus \mathcal{B}$. By definition of $\mathcal{B}$, there exist constants $\epsilon > 0$ and $S > 0$ such that

$$\mathbf{E}_x u(X^\eta_s) > \epsilon s, \quad \text{for all } s \ge S. \tag{3.25}$$

The second statement in (3.19) implies that $|u(x) - u(0)| \leq (q_u + q_d)|x|$. From this, it follows

$$c(x) \geq Au(x) - B,$$

for $A = \alpha/(q_u + q_d)$ and $B = \alpha u(0)/(q_u + q_d) + K$, see (3.4). In view of (3.25), this implies

$$\limsup_{T\to\infty} \frac{1}{T}\mathbf{E}_x\left(\int_0^T c(X_s^\eta)ds\right) \geq \limsup_{T\to\infty} \frac{1}{T}\int_S^T (A\epsilon s - B)ds = \infty.$$

As a consequence, for any $\eta \in \mathcal{A} \setminus \mathcal{B}$, we have

$$\limsup_{T\to\infty} \frac{1}{T}\mathbf{E}_x\left(\int_0^T c(X_s^\eta)ds + q_u U_T + q_d D_T\right) = \infty.$$

Finally, as the class of reflecting controls gives finite ergodic limits by Lemma 3.7, the infimum can be taken in the subclass $\mathcal{B}$. So Theorem 3.4.1 gives the equality $G(x) = \lambda = C(a, b)$ (see Theorem 3.3.5), concluding the proof. □

3.5 Examples

We present some examples that are solved by using Theorem 3.3.5. We remark that the examples are similar to the ones showed in [Alvarez(2018)].

3.5.1 Orstein–Uhlenbeck process

The cost function now has the form

$$c(x) = \max(-\alpha x, x), \quad \alpha > 0, \tag{3.26}$$

We consider a mean reverting process that follows the stochastic differential equation

$$dX_t = -\theta X_t dt + \sigma dW_t, \ \sigma, \theta > 0. \tag{3.27}$$

To satisfy Assumptions 3.2.1 , we add the restrictions $q_d\theta < 1$ and $q_u\theta < \alpha$ and take $x_1^0, x_2^0 = 0$. In this case

$$S'(x) = e^{\frac{\theta x^2}{\sigma^2}}, \qquad m'(x) = \frac{2}{\sigma^2}e^{\frac{-\theta x^2}{\sigma^2}},$$

and the function $a \to b_a$ defined in (3.12) is

$$b_a = a\left(\frac{-\alpha + q_u\theta}{1 - q_d\theta}\right). \tag{3.28}$$

Using the Corollary 3.3.2 we deduce that what is left to do is to find a root a^* of the function

$$I_2(a, b_a) = \int_a^{b_a} (\pi_2(t) - \pi_2(a))\, m'(t)dt + \frac{q_u + q_d}{S'(b_a)}, \;\; a \in (-\infty, 0]$$

We separate in terms:

$$\begin{aligned}
\int_a^{b_a} \pi_2(t)m'(t)dt &= \int_a^0 \frac{2t(-\alpha + q_u\theta)}{\sigma^2} e^{\frac{-\theta t^2}{\sigma^2}} dt + \int_0^{b_a} \frac{2t(1 + q_u\theta)}{\sigma^2} e^{\frac{-\theta t^2}{\sigma^2}} dt \\
&= \int_{\frac{a^2}{\sigma^2}}^0 (-\alpha + q_u\theta)e^{-\theta u}du + \int_0^{\frac{b_a^2}{\sigma^2}} (1 + q_u\theta)e^{-\theta u}du \\
&= \frac{1}{\theta}\left((-\alpha + q_u\theta)(e^{-\theta\frac{a^2}{\sigma^2}} - 1) + (1 + q_u\theta)(1 - e^{-\theta\frac{b_a^2}{\sigma^2}})\right).
\end{aligned} \tag{3.29}$$

On the other hand:

$$\begin{aligned}
&\int_a^{b_a} -\pi_2(a)m'(t)dt + \frac{q_u + q_d}{S'(b_a)} \\
&\qquad = a(\alpha - q_u\theta)\sqrt{\frac{\pi}{\sigma^2\theta}}\left(erf\left(\sqrt{\frac{\theta}{\sigma^2}}b_a\right) - erf\left(\sqrt{\frac{\theta}{\sigma^2}}a\right)\right) + (q_u + q_d)e^{\frac{-\theta b_a^2}{\sigma^2}}.
\end{aligned} \tag{3.30}$$

From equations (3.28), (3.30) and (3.30) we deduce the barriers $(a^*, b^*) \in (-\infty, 0) \times (0, \infty)$ that define the optimal control are defined by the equations:

$$\begin{aligned}
&\frac{1}{\theta}\left((-\alpha + q_u\theta)(e^{-\theta\frac{(a^*)^2}{\sigma^2}} - 1) + (1 + q_u\theta)(1 - e^{-\theta\frac{(b^*)^2}{\sigma^2}})\right) \\
&\qquad + a^*(\alpha - q_u\theta)\sqrt{\frac{\pi}{\sigma^2\theta}}\left(erf\left(\sqrt{\frac{\theta}{\sigma^2}}b^*\right) - erf\left(\sqrt{\frac{\theta}{\sigma^2}}a^*\right)\right) + (q_u + q_d)e^{\frac{-\theta(b^*)^2}{\sigma^2}} = 0.
\end{aligned}$$

and

$$b^* = a^*\left(\frac{-\alpha + q_u\theta}{1 - q_d\theta}\right).$$

Moreover using (3.17), we get $G(x) = \lambda^*$ is equal to $a^*(-\alpha + q_u\theta)$. For example if we take the parameters $\theta = 0.1,\ \alpha = 2,\ \sigma = 3,\ q_u = 3,\ q_d = 1$, we get $a^* = -3.100,\ b^* = 5.856,\ \lambda^* = 5.270$.

3.5.2 Brownian motion with drift

We present an easier to compute example to give a more explicit representation of the optimal reflecting control. The cost function now is $c(x) = |x|$. The underlying process is a Brownian motion with drift μ and volatility $\sigma \neq 0$, thus

$$dX_t = \mu t + \sigma dW_t, \ \{W_t\}_{t\geq 0} \text{ a standard Brownian motion.}$$

Assumptions 3.2.1 , are satisfied automatically by taking $x_1^0, x_2^0 = 0$. First, we assume $\mu \neq 0$. In this case

$$S'(x) = e^{\frac{-2\mu x}{\sigma^2}}, \qquad m'(x) = \frac{2}{\sigma^2} e^{\frac{2\mu x}{\sigma^2}},$$

and the function $a \to b_a$ defined in (3.12) is

$$b_a = -a - \mu(q_u + q_d). \tag{3.31}$$

Again, from Corollary 3.3.2 we need to find the root of the function $I_2(a, b_a), \ a \in (-\infty, 0)$. In this case the equation to solve is

$$I_2(a, b_a) = \int_a^{b_a} \frac{2}{\sigma^2} \left(|t| + a\right) e^{\frac{2\mu t}{\sigma^2}} dt + (q_u + q_d) e^{\frac{2\mu b_a}{\sigma^2}} = 0. \tag{3.32}$$

We proceed with the calculations and rewrite the integral:

$$\int_a^0 -\frac{2}{\sigma^2} t e^{\frac{2\mu t}{\sigma^2}} dt + \int_0^{b_a} \frac{2}{\sigma^2} t e^{\frac{2\mu t}{\sigma^2}} dt + \frac{a(e^{\frac{2\mu b_a}{\sigma^2}} - e^{\frac{2\mu a}{\sigma^2}})}{\mu} + (q_u + q_d) e^{\frac{2\mu b_a}{\sigma^2}} = 0.$$

By substitution, we get:

$$-\frac{1}{\mu} \int_{\frac{2\mu a}{\sigma^2}}^0 u e^u du + \frac{1}{\mu} \int_0^{\frac{2\mu b_a}{\sigma^2}} u e^u du + \frac{a(e^{\frac{2\mu b_a}{\sigma^2}} - e^{\frac{2\mu a}{\sigma^2}})}{\mu} + (q_u + q_d) e^{\frac{2\mu b_a}{\sigma^2}} = 0.$$

Taking primitives:

$$\frac{1}{\mu}\left(2 + e^{\frac{2\mu a}{\sigma^2}} \left(\frac{2\mu a}{\sigma^2} - 1\right) + e^{\frac{2\mu b_a}{\sigma^2}} \left(\frac{2\mu b_a}{\sigma^2} - 1\right)\right) + \frac{a(e^{\frac{2\mu b_a}{\sigma^2}} - e^{\frac{2\mu a}{\sigma^2}})}{\mu} + (q_u + q_d) e^{\frac{2\mu b_a}{\sigma^2}} = 0.$$

Multiplying the equation by μ and using (3.31), we deduce equation (3.32) is equivalent to

$$e^{\frac{2\mu a}{\sigma^2}}\left(\frac{2\mu a}{\sigma^2}-1-a\right)+e^{\frac{2\mu}{\sigma^2}(-a-\mu(q_u+q_d))}\left(\frac{2\mu(-a-\mu(q_u+q_d))}{\sigma^2}-1+a+\mu(q_u+q_d)\right)+2=0. \tag{3.33}$$

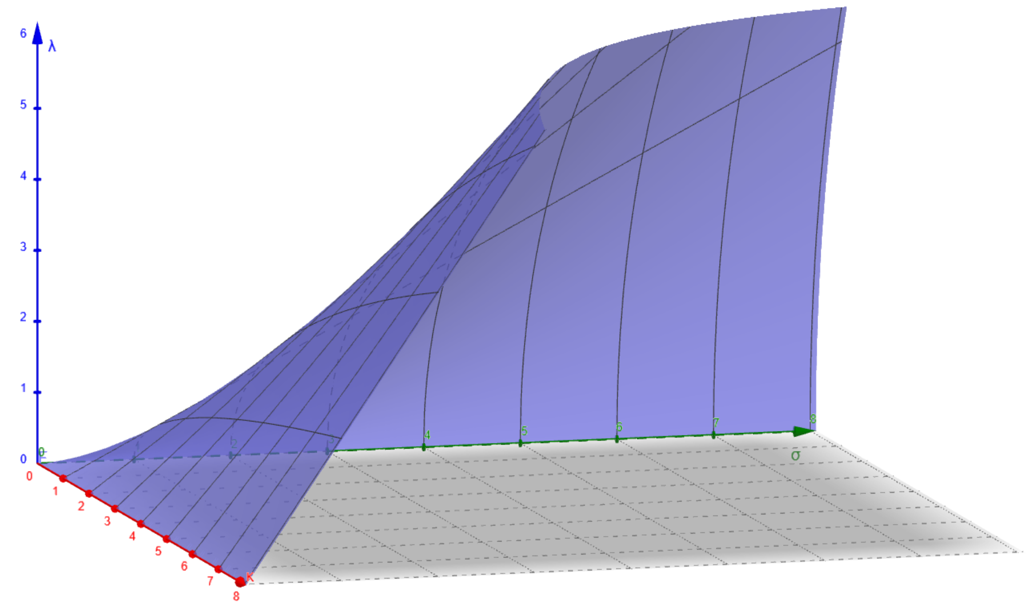
With equations (3.17), (3.31) and (3.33) the problem is characterized as a root of a function of one variable. For example, when $\sigma = 2,\ \mu = -1,\ q_u = 0.4,\ q_d = 0.6$ we get $a^* = -0.417,\ b^* = 1.417,\ \lambda^* = 0.817$.

In the case $\mu = 0$, by standard integration we obtain

$$-a^* = b^* = \lambda^* = \sqrt{\frac{(q_u+q_d)\sigma^2}{2}}.$$

For this case, denoting $K := q_u + q_d$, we present a graph of the function

$$\lambda(\sigma, K) = \sqrt{\frac{(q_u+q_d)\sigma^2}{2}}.$$



[1]

[1]The graph was made with the software Geogebra, see [Geogebra (2024)].

Chapter 4

Two-sided ergodic singular mean field games for Itô-diffusions

Abstract

In this chapter, we study a probabilistic mean field game driven by a linear diffusion. To be more precise, an individual player aims to minimize an ergodic long-run cost by controlling the diffusion through a pair of –increasing and decreasing– càdlàg processes, while interacting with an aggregate of players through the expectation of a similar diffusion controlled by another pair of càdlàg processes. We consider the control problem formulated in Chapter 3 with an added pool of players whose controls are reflecting controls and study the existence and uniqueness of equilibrium points in this game. Furthermore, we examine the convergence of a finite-player game to this problem to justify our approach.

We study the problem posed in 2.9 when the underlying process is under the same hypothesis as the previous chapter (see equation (3.2)). Apart from the first section, all of this chapter is based on [Christensen et al. (2023)].

This chapter is organized as follows. In Section 4.1 we give some historical remarks and an introduction of stationary mean field games. In Section 4.2 we give the framework of the chapter. In Section 4.3 we consider the mean field game problem. It adds the complexity of a two-variable cost function where the second variable represents the market. The main result consists of a set of conditions for the existence and uniqueness of equilibrium controls, containing also a particular analysis when the cost function is multiplicative. Section 4.4 presents three examples that illustrate these results. Section 4.5 contains approximation results. The equilibrium found for mean field games, becomes the limit of Nash equilibrium controls when considering an individual player in the framework of a symmetric N-player game.

4.1 Introduction

In recent years, mean field game theory has emerged as a powerful framework for modeling the behavior of large populations of interacting players in a stochastic environment. This interdisciplinary field lies at the intersection of mathematics, economics, and engineering, offering deep insights into complex systems characterized by strategic interactions. Mean field game models have found applications in various domains, including for instance, finance, energy systems [Carmona(2021)], or traffic management and social dynamics [Festa and Göttlich (2018)]. The two seminal papers in the field can be considered the contributions by [Huang et al. (2006)] and [Lasry and Lions (2007)]. The key issue in their proposals, under the assumption of a large number of identically interacting players, is that individual actions do not affect a mean state of the system. This means that an individual player faces an optimization problem against a synthetic player, resulting from the aggregation of a large number of players, which is referred to in this chapter as *the market*. The success of the proposal made it possible to solve various problems, many of which can be found in the two-volume monograph by [Carmona and Delarue (2018)], which has become a central reference in the field.

Our aim in this chapter is to incorporate a mean field game dependence into the two-sided ergodic singular control problem for Itô-diffusions described in Chapter 3. As a consequence, we obtain necessary and sufficient conditions for the existence of mean field game equilibrium points, and, for more restricted families of cost functions, uniqueness within the class of reflecting controls. Finally, we define an N-player problem and prove that a mean field equilibrium is an approximate Nash equilibrium for the N-player game.

The mean field game framework is less discussed in the literature. However, there has been increased activity in this area in the recent past. Here we would like to mention papers that study similar problems that the one posed in this chapter.

- [Carmona, et. al. (2013)] Although this paper has structural differences with our problem, we mention it because it has greatly contributed to the development of mean field games. The underlying diffusion represents the log-monetary reserves of banks lending to and borrowing from other. The controls α_t^i are through the drift which represent the rate of exchange of the banks. The SDE is of the form

$$dX_t^i = \frac{a}{n}\sum_{j=1}^{N}(X_t^j - X_t)dt + \alpha_t^i dt + \sigma W_t^i + \sigma\sqrt{1-\rho^2}W_t^0, \quad i = 1, \dots, N,$$

  where $W^i,\ i = 0, \dots, N$ are independent Brownian motions, called *idiosyncratic noise*, which represent the *randomness* that affects individual components of the system independently of one another and W^0 represents a common noise that affects all the system simultaneously. All the parameters are positive and the objective of each agent is to minimize a finite time horizon integral cost and a terminal cost, both penalizing the deviation from the empirical mean and the running cost also penalizing the value α_t^2. Different types of Nash equilibrium (see 1.2.3) are obtained using the Pontryagin principle, a FBSDE approach and a HJB approach (see [Carmona and Delarue (2018), Chapter II]). The mean field game equilibrium is also obtained explicitly using a HJB approach and the convergence from finite many agents to the mean field game is proved. The authors remark that although the Nash Equilibrium is explicitly obtained, the usefulness of the study of the mean field game lies in the fact that it is a problem less affected by small perturbations.

- [Lacker (2015)] The author, under general assumptions, proves the existence of a *relaxed mean field equilibrium* when the underlying process is an Itô-diffusion and the controls are exercised over the drift and the volatility. We proceed to give an informal explanation of this notion of equilibrium (in fact the notation of the article is simplified here). Under the usual framework for finite-horizon mean field game problems for d- dimensional Itó diffusions, there is a flow of probability measures $\mu = \{\mu_t\}_{t\geq 0}$ in $\mathbb{R}^d$ that represents the aggregate of players, a control $\alpha = \{\alpha_t\}_{t\geq 0}$ and a controlled vector process X^α satisfying the SDE:

$$dX_t^{i,\alpha} = b_i(t, X_t, \mu, \alpha_t)dt + \sigma_i(t, X_t, \mu_t, \alpha_t)dW_t.$$

There are a couple of functions f, g representing a running cost/reward and terminal cost/reward respectively. The objective is to find a control α^* such that

$$\alpha^* \in \arg\max_\alpha \{J(\alpha, \mathbf{P}(X_t^{\alpha^*} \in dy))\},$$

with

$$J(\alpha, \mu) = \mathbf{E}\left(\int_0^T f(t, X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right).$$

In the *relaxed mean field game* the controls, called *relaxed controls*, are random measures q on $[0, T] \times A$ (A the control space) such that for $0 \leq s \leq t \leq T$, $q([s, t], A)(\omega) = t - s$ and some more technical integrability conditions. The controlled process X^q is defined in a weak sense with its infinitesimal generator such that the measure $\mathbf{P}(X_t^q \in dy)$ is well defined. The *relaxed mean field equilibrium* is a *relaxed control* q^* that satisfies

$$q^* \in \arg\max_q \{J(q, \mu)\},$$

with

$$J(q,\mu) = \mathbf{E}\left(\int_0^T f(t, X_t^q, \mu_t, \alpha_t)dt + g(X_T^q, \mu_T)\right).$$

The main results are a theorem that guarantees the existence of *relaxed mean field equilibrium* (using a fixed point theorem) and under stricter hypotheses, the existence of a mean field equilibrium (in the usual sense).

- [Fu and Ulrich (2017)] In the problem posed in this article, the author works with regular controls (in the drift and volatility) and singular controls. To be more precise, the MFG consists in finding a couple of controls process α^*, Z^* (the first is the regular and the second is the singular) such that

$$\{\mu_t\}_{t\geq 0} = \{\mathbf{P}(X_t^{\alpha^*,Z^*} \in dy)\}_{t\geq 0},$$
$$(\alpha^*, Z^*) \in \arg\min_{\alpha,Z} \mathbf{E}\left(\int_0^T f(t, X_t^{\alpha,Z}, \mu_t, \alpha_t)dt + g(X_T^{\alpha,Z}, \mu_T) + \int_0^T h(t)dZ_t\right),$$
$$dX_t^{\alpha,Z} = b(t, X_t^{\alpha,Z}, \mu_t, \alpha_t)dt + \sigma(t, X_t^{\alpha,Z}, \mu_t, \alpha_t)dW_t + c(t)dZ_t.$$

Existence of a *relaxed MFG equilibrium* is proved under general hypotheses (by using a fixed point theorem). Moreover, by taking adequate controls and stricter hypotheses the author showed that a problem with regular controls can approximate one with singular ones. We remark that Z_t is assumed to be non-decreasing.

- [Lacker and Zariphopoulou (2019)] The authors study two problems applied to finance (see CARA and CRRA utilities in [Back (2017), Chapter I]). They are finite horizon problems that only take into account the terminal cost at time T. In each one the N-player game and the MFG are studied, conditions are given for existence and uniqueness of constant equilibriums (they are given explicitly and convergence is proved) using a HJB equation. Without going into technical details, both processes are controlled Itô-diffusions. In the first case, the process X^i represent the wealth of player i and the controls represent the amount invested in the stock. The objective function penalizes deviating from a factor of the empirical mean of the wealth of all players (this parameter is called *competition weight*). In the second problem, the controls are treated as fractions of wealth that the player i invests in the stock at time t. The objective functions take into account the relative wealth with respect to the empirical mean at time T.
- [Guo and Renyuan (2019)] The authors study a two-sided singular discounted mean field game and N-player problems. The integral running cost function is of the form $h(x_i - m)$ with m representing the empirical mean and x_i the state of player i (deviating from the mean is penalized). The function h is convex, with $0 < k < h'' < K$. The underlying

process is a Brownian motion. The problem is also known as *fuel follower problem*. The authors give an explicit Nash equilibrium control (that depends on the state of the players and is not unique) and a MFG equilibrium reflecting control. They also prove that the MFG equilibrium reflecting control is an ϵ_N-Nash equilibrium for the N-player problem with $\epsilon_N = O(\frac{1}{\sqrt{N}})$.

- [Christensen et al. (2021)] The authors leave the classical setting of continuous stochastic control and consider an ergodic *stochastic impulse control problem*. To be more specific, they study a continuous time model with interventions at adaptively chosen discrete time only. These controls arise whenever a cost $K > 0$ have to be paid for each intervention. In this case the problem is related to resource management (harvesting to be more specific). The *admissible controls* are $R = \{\tau_n\}_{n\in\mathbb{N}}$, with each τ_n a stopping time such that $\tau_n < \tau_{n+1}$. At each time τ_n, the controlled process, denoted X^R, is taken back to zero. The underlying process is an Itô-diffusion and the controlled process satisfies the equation:

$$X_t^R = X_0^R + \int_0^t \mu(X_s^R)ds + \int_0^t \sigma(X_s^R)dWs - \sum_{\tau_n \leq t}(X_{\tau_n^-} - x).$$

In this optimization problem, the controlled process represents the forest stand and the running cost function is the price of the wood. To define the mean field game, the flow of players are represented with the controlled process X^Q and the associated stopping times σ_n are the same in law once shifted. The authors propose two kinds of dependence on the flow of players. In the first one, the wood price depends on the average harvesting rate. Thus, using regenerative theory, the reward function is of the form

$$J_x(R,Q) = \liminf_{T\to\infty} \mathbf{E}_x \left(\sum_{\tau_n \leq T} \left(\gamma \left(X_{\tau_n^-}^R, \frac{\mathbf{E}_x(X_{\sigma^-}^Q) - x}{\mathbf{E}(\sigma)} \right) - K \right) \right).$$

In the second case, they assume that the prices do not depend on the average harvesting rate, but on the expected wood supply thus the reward function is

$$J_x(R,Q) = \liminf_{T\to\infty} \mathbf{E}_x \left(\sum_{\tau_n \leq T} \left(\gamma \left(X_{\tau_n^-}^R, \mathbf{E}_x(X_\infty^Q) - x \right) - K \right) \right).$$

in both cases, they provide criteria to guarantee the existence of MFG equilibrium controls. But general conditions for uniqueness are only obtained in the first case. Moreover they also study the problem where the agents cooperate, that is to find Q satisfying

$$J(Q,Q) = \sup_R J(R,R).$$

We want to remark that this problem has an adjoint optimal stopping problem and the verification theorems are not presented in the form of a HJB equation but as a more probabilistic formulation relying on martingale's theory.

- [Cao and Guo(2022)] This paper analyzes a class of MFGs, when the underlying process is a geometric brownian motion with singular controls. The running cost function is of the form $f(x,\mu) = x^\alpha \rho(\mu)$. The problem is similar to 2.7.2. The authors provide an explicit solution to the MFG, they present a sensitivity analysis to compare the solution to the MFG with that of the single-agent control problem, and establish its approximation to the corresponding N-player game in the sense of ϵ_N, with $\epsilon_N = O\left(\sqrt{\frac{1}{N}}\right)$. A HJB equation is used to obtain the solution of the control problem and a fixed point theorem to get a MFG equilibrium control (under given hypotheses).
- [Aïd et al.(2023)] The problem posed in this paper represents a continuous firm of unitary mass indexed by their production capacity. Firms behave competitively and the company can force productive shocks. the economy can vary between two states. The state that is the economy at time t is represented by the process $\epsilon_t \in \{1,2\}$. There are two parameters $p_1, p_2 \in (0,1)$. If the economy is in state i, it stays in that state an exponential time with parameter p_i (there is independence between the waiting times and the associated Brownian motion $W = \{W_t\}_{t\geq 0}$). The company is allowed to give a productivity shock at any time represented by the increasing control $I = \{I_t\}_{t\geq 0}$. The controlled process representing the productivity of the company is then

$$X_t^I = x \exp\left(-\left(\delta t + \frac{1}{2}\int_0^t \sigma_{\epsilon_s}^2 ds \right) + \int_0^t \sigma_{\epsilon_s} dW_s + I_t \right), \ \delta, \sigma_i > 0.$$

When

  - the process starts at the point x,
  - the economy departs in the state i and
  - the control representing the flow of the players is Q_t,

the profit of the company is $J_{(x,i)}(I,Q)$. The problem is discounted with infinite-time horizon. Without delving into much detail, its running function depends on $\eta_{\epsilon_t}(Q)X_t^I$, each function η_i depending on the stationary distribution on Q, plus $-c(X_t^l)^2$, $c>0$ and its singular part penalizes the controls. We remark, that the discount factor should be bigger than $2\max\{\sigma_1^2, \sigma_2^2\}$, thus it is not clear that the abelian limit holds. To solve the problem, two HJB equations arise, each one with an associated optimal stopping problem (depending on the state of the economy). For a fixed $Q = (Q_1, Q_2) \in \mathbb{R}^2_+$, the optimal controls are given by two barriers a_1^Q, a_2^Q such that in the state i the controls is the reflection in (a_i, ∞) (this control pushes the process to a_i if the process is below a_i when the state of the economy is i). Existence and uniqueness of MFG equilibrium are proved using a fixed point theorem.

- [Cao et al.(2023)] The authors study an ergodic and discounted singular MFG. The underlying process is an Itô-diffusion and the control is non-decreasing (obviously necessary hypotheses are given so that the problem does not degenerates). The control problems are similar to 2.7.2 and 2.7.1 but in this case, both are one sided, so in both control problems a reflection in a half-line is the optimal control. To solve the discounted control problem, a HJB equation is solved and the discounted MFG is characterized with the unique root of an equation. The ergodic control problem follows a different path. The authors, informally speaking, differentiate the infinitesimal generator in the adjoint HJB equation, thus obtaining a new free boundary problem that is associated to an optimal stopping problem with a different underlying process (this is due to the fact that μ and σ are functions affected by differentiation). That way, they solve the new HJB equation and by taking an adequate primitive they solve the original HJB equation. Again the MFG equilibrium is obtained as the unique root of an equation.
  The abelian limit is obtaining by proving that the map that associated the discount to the extreme of the half-line that defines the optimal control is a continuous function. Finally, the authors study the relationship of the MFGs with the N-player problem.

4.2 Setting

The probability space, the underlying diffusion and the admissible controls are the same as the ones defined in 3.2. Due to the presence of a flow of players we need to change the cost function.

We introduce below the cost function $c(x, y)$ to be considered in the mean field game formulation, satisfying some natural conditions.

Assumption 4.2.1. *Assume that $c\colon \mathbb{R}^2 \to \mathbb{R}_+$ is a continuous function, and the positive constants q_u, q_d are the unit cost of using the associated controls. Assume that, for each fixed $y \in \mathbb{R}$ there exist a value x_y such that*

$$c(x, y) \geq c(x_y, y) \geq 0, \qquad \textit{for all } x \in \mathbb{R},$$

and constants $K_y \geq 0$ and $\alpha_y > 0$ such that

$$c(x, y) + K_y \geq \alpha_y |x|, \qquad \textit{for all } x \in \mathbb{R}. \tag{4.1}$$

Consider the maps

$$\pi_1(x, y) = c(x, y) + q_d \mu(x), \qquad \pi_2(x, y) = c(x, y) - q_u \mu(x),$$

and assume that for each fixed $y \in \mathbb{R}$:

(i) *There exists a unique real number $x_i^y = \arg\min\{\pi_i(x, y) \colon x \in \mathbb{R}\}$ so that $\pi_i(\cdot, y)$ is decreasing on $(-\infty, x_i^y)$ and increasing on (x_i^y, ∞), where $i = 1, 2$.*

(ii) *The following limits hold:*

$$\lim_{x \to \infty} \pi_1(x, y) = \lim_{x \to -\infty} \pi_2(x, y) = \infty. \tag{4.2}$$

4.3 Characterization of the MFG equilibrium

The study of the existence and uniqueness of equilibrium points begins with the application of Theorem 3.4.2 when the state of the market is asymptotically constant. The cost function becomes one-dimensional and the results in [Alvarez(2018)] can be applied.

More precisely, assuming $f(x)$ continuous, the expectation of the market diffusion $\mathbf{E}_x(f(X_t^{c,d}))$ has an ergodic limit, denoted $R(c, d)$, and applying the previous results, we can prove that the optimal controls for the player should be found in the class of reflecting controls, considering a one variable cost function of the form $c(\cdot, R(c, d))$. This is why we assume that the market is also controlled by reflections at some levels $c < d$, and expect to obtain an equilibrium point when the optimal levels $a < b$ that control the player's diffusion coincide with $c < d$ (see Definition 4.3.1). Note that the question of the existence of equilibrium controls beyond the class of reflecting controls is not addressed here. The requirements to apply these results in the mean field game formulation follow.

4.3.1 Conditions for optimality and equilibrium

In this setting, we can generalize the results of the section before using some simple ergodic results for diffusions. Recall that the function $f(x)$ is assumed to be continuous.

Definition 4.3.1. *We say that a control η^* is an* equilibrium *of the mean field game if it belongs to the set*

$$\underset{\eta=(U,D)\in\mathcal{A}}{\arg\min}\left\{\limsup_{T\to\infty}\frac{1}{T}\mathbf{E}_x\left(\int_0^T c\big(X_s^\eta,\mathbf{E}_x(f(X_s^{\eta^*}))\big)ds+q_uU_T+q_dD_T\right)\right\}.$$

In case the control is reflecting, i.e. $\eta^*=(U^{a^*,b^*},D^{a^*,b^*})$ *we say that* (a^*,b^*) *is an* equilibrium point*.*

The idea of the above definition is to consider situations in which the individual player has no incentive to act differently to the market. Regarding the three-step proposal of [Carmona and Delarue(2013), Section 2.2], we would (i) choose a control $\mu\in\mathcal{A}$ for the market, (ii) solve the standard stochastic problem

$$\inf_{\eta=(U,D)\in\mathcal{A}}\left\{\limsup_{T\to\infty}\frac{1}{T}\mathbf{E}_x\left(\int_0^T c\big(X_s^\eta,\mathbf{E}_x(f(X_s^{\mu}))\big)ds+q_uU_T+q_dD_T\right)\right\}.$$

to obtain a control η (depending on μ), and (iii) find a fixed point in $\mathcal{A}$ of the map $\mu\to\eta$. Compared to [Cao et al.(2023), Definition 3.2], closer to our formulation, Definition 4.3.1 admits a time dependent value representing the market state. More precisely, in [Cao et al.(2023)], the authors consider situations in which the controlled market process has a stationary distribution, whose mean has to coincide with the equilibrium value. If this is the case, the control to be an equilibrium, in general terms, should be a reflecting one. Nevertheless, as the following results shows, when considering reflecting controls, we can substitute the time dependent value by its limit in Definition 4.3.1.

Theorem 4.3.1. *Consider the points $a<b$, $c<d$, and $x\in\mathbb{R}$. Then*

$$\begin{aligned}\limsup_{T\to\infty}\frac{1}{T}\mathbf{E}_x&\left(\int_0^T c(X_s^{a,b},\mathbf{E}_x\big(f(X_s^{c,d})\big))ds+q_d dD_s^{a,b}+q_u dU_s^{a,b}\right)\\&=\frac{1}{m(a,b)}\left(\int_a^b c(u,R(c,d))m(du)+\frac{q_u}{S'(a)}+\frac{q_d}{S'(b)}\right),\end{aligned}\tag{4.3}$$

where

$$R(c,d)=\int_c^d\frac{f(u)}{m(c,d)}m(du).$$

Proof. Applying Lemma 3.3.1 with the cost function $c(\cdot,R(c,d))$ we obtain that

$$\lim_{T\to\infty} \frac{1}{T}\mathbf{E}_x \left(\int_0^T c(X_s^{a,b}, R(c,d))ds + q_u U_T^{a,b} + q_d D_T^{a,b} \right)$$
$$= \frac{1}{m(a,b)} \left(\int_a^b c(u, R(c,d))m(du) + \frac{q_u}{S'(a)} + \frac{q_d}{S'(b)} \right),$$

i.e. the r.h.s. in (4.3). It remains then to verify that

$$\limsup_{T\to\infty} \frac{1}{T}\mathbf{E}_x \left(\int_0^T |c(X_s^{a,b}, \mathbf{E}_x(f(X_s^{c,d}))) - c(X_s^{a,b}, R(c,d))|ds \right) = 0. \tag{4.4}$$

In order to do this, define the continuous function $H\colon f([c,d]) \to \mathbb{R}^+$ by

$$H(y) = \max_{u\in[a,b]} |c(u,y) - c(u, R(c,d))|,$$

and observe that the limit in (4.4) can be bounded by

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T H(\mathbf{E}_x(f(X_s^{c,d})))ds = \limsup_{T\to\infty} \frac{1}{T} \int_0^T H\left(\int_c^d f(y)\mathbf{P}_s(x,dy) \right) ds,$$

with $\mathbf{P}_s(x,dy) = \mathbf{P}_x(Y_s^{c,d} \in dy)$. This limit is zero because

$$H\left(\int_c^d f(y)\mathbf{P}_s(x,dy) \right) \to H(R(c,d)) = 0,$$

as H is uniformly continuous, bounded and

$$\left\| \mathbf{P}_s(x,\cdot) - \frac{1}{m(c,d)} m(\cdot) \right\| \to 0, \qquad \text{as } s\to\infty,$$

with the norm of total variation (see [Rogers and Williams (2000), Theorem 54.5]). It follows that (4.4) holds, concluding the proof. □

The existence and uniqueness of minimizers given in (b) in Corollary 3.3.2 can also be generalized, by noticing that in Theorem 4.3.1 the second variable in the cost function is fixed. The optimality of reflecting controls within the class of càdlàg controls corresponding to Definition 4.3.1 follows from Theorem 3.4.2.

Theorem 4.3.2. *For a fixed (a,b), the infimum of the ergodic problem is reached only at a pair (a^*, b^*) such that*

(i) $\pi_1(b^*, R(a,b)) = \pi_2(a^*, R(a,b))$,

(ii) $\displaystyle\int_{a^*}^{b^*} (\pi_1(t, R(a,b)) - \pi_1(b^*, R(a,b)))\, m(dt) + \frac{q_u + q_d}{S'(a^*)} = 0.$

Moreover $(a^*, b^*) \in (-\infty, x_2^{R(a,b)}) \times (x_1^{R(a,b)}, \infty)$

Based on this result we obtain a condition for equilibrium of the mean field game (see Definition 4.3.1).

Theorem 4.3.3. *A pair $a < b$ is an equilibrium point if and only if*

(i) $\pi_1(b, R(a,b)) = \pi_2(a, R(a,b))$,

(ii) $\displaystyle\int_{a}^{b} (\pi_1(t, R(a,b)) - \pi_1(b, R(a,b)))\, m(dt) + \frac{q_u + q_d}{S'(a)} = 0.$

Moreover $(a, b) \in (-\infty, x_2^{R(a,b)}) \times (x_1^{R(a,b)}, \infty)$.

4.3.2 The multiplicative case

In this subsection, we assume that the cost function has a multiplicative form.

Assumption 4.3.4. *The cost function satisfying Assumption 4.2.1, is factorized as*

$$c(x, y) = g(x)h(y),$$

where the factors satisfy

(i) $g\colon \mathbb{R} \to [0, \infty)$ *is a convex function, with* $g(x) \geq g(0)$,

(ii) $h\colon \mathbb{R} \to (0, \infty)$ *is continuous, with* $h(x) \geq h(0)$.

Note that such a multiplicative decomposition can be applied when $g(x)$ is interpreted as a standardized representation of the units of a good corresponding to a state x and $h(y)$ as the factor that models the market-based unit cost of maintenance, as an example in forestry of $g(x)$ modeling the forest stand, see [Sohr, T. (2020), 5.2.1].

We give a first result that follows from Theorem 4.3.3 if the cost function is multiplicative. In this situation, using condition (i), one of the variables can be obtained as a function of the other. For this purpose, consider the set

$$C_a = \{b \in \mathbb{R}\colon b > x_1^{R(a,b)} \vee a,\ x_2^{R(a,b)} > a,\ \pi_1(b, R(a,b)) = \pi_2(a, R(a,b))\}.$$

Observe that if $C_a = \emptyset$, there are no equilibrium points. We then assume condition $C_a \neq \emptyset$ if and only if $a \leq 0$. This means that we search for the equilibrium points in a connected set.

Furthermore, for a fixed $a \leq 0$ we denote

$$\rho(a) = \inf C_a, \tag{4.5}$$

and

$$L(a) = R(a, \rho(a)).$$

Proposition 4.3.5. *Suppose that the cost function factorizes as in Assumption 4.3.4, and there exists a point $a_0 \leq 0$ such that the function ρ defined via (4.5) is continuous in $(-\infty, a_0]$. Then,*

(C_1) *if*

$$\int_{a_0}^{\rho(a_0)} (\pi_1(t, L(a_0)) - \pi_1(\rho(a_0), L(a_0)))m(dt) + \frac{q_u + q_d}{S'(a_0)} \geq 0,$$

then there is at least one equilibrium point.

(C_2) *Furthermore, if in $(-\infty, a_0]$,*

$$\pi_2(t, L(a_2)) - \pi_2(a_2, L(a_2)) < \pi_2(t, L(a_1)) - \pi_2(a_1, L(a_1)) \qquad \forall (a_2, a_1, t) \quad s.t, \; a_2 < a_1 < t \leq a_0,$$

$$\pi_1(t, L(a_2)) - \pi_1(\rho(a_2), L(a_2)) < \pi_1(t, L(a_1)) - \pi_1(\rho(a_1), L(a_1)) \qquad \forall (a_2, a_1, t) \quad s.t. \; \rho(a_2) > \rho(a_1) > t \geq a_0,$$

and

$$\int_r^l (\pi_1(t, R(r,l)) - \pi_1(l, R(r,l)))m(dt) + \frac{q_u + q_d}{S'(r)} > 0, \qquad \forall r \in (a_0, \rho(a_0)), l > r, \; \pi_1(l, R(r,l)) = \pi_2(r, R(r,l)), \tag{4.6}$$

then the equilibrium is unique.

Proof. For the existence of equilibrium points, we need to prove

$$\int_A^{\rho(A)} (\pi_1(t, L(A)) - \pi_1(\rho(A), L(A)))m(dt) + \frac{q_u + q_d}{S'(A)} < 0,$$

for some $A < a_0$. First, observe that the inequality can be rewritten as

$$\int_A^0 (\pi_2(t, L(A)) - \pi_2(A, L(A)))m(dt) \\ + \int_0^{\rho(A)} (\pi_1(t, L(A)) - \pi_1(\rho(A), L(A)))m(dt) + \frac{q_u + q_d}{S'(0)} < 0. \quad (4.7)$$

Furthermore, due to the nature of the multiplicative cost, the points $x_i^y, i = 1, 2$ defined in (4.2.1) can be taken all equal to x_i^0 for each i respectively. Thus, for A negative enough, both integrands are always negative and tend to $-\infty$ when $A \to -\infty$. Finally, for the uniqueness, condition (C_2) implies that the map defined in $(-\infty, a_0]$:

$$a \to \int_a^{\rho(a)} (\pi_1(t, L(a)) - \pi_1(\rho(a), L(a)))m(dt) + \frac{q_u + q_d}{S'(a)},$$

is monotone, thus concluding that the root of this map is unique. □

Remark 4.3.1. *Condition (C_2) is a condition on differences of value functions. In particular, if we assume $\pi_2 \in C^2((-\infty, a_0) \times \mathbb{R})$, f defined in the introduction of the section is increasing and $L(a)$ is increasing, then the first inequality in condition (C_2) holds if π_2 has negative cross second derivative in $(-\infty, a_0) \times \mathbb{R}$ which is equivalent to the function*

$$(a, \mu) \to \pi_2(a, \langle f, \mu \rangle), \quad a \in (-\infty, a_0), \ \mu \text{ a probability measure},$$

being submodular (see [Dianetti et. al. (2019), Assumption 2.9 and Example 2]). A similar analysis can be made with the second inequality (the function in this case is supermodular).

In the particular case of a diffusion without drift, the conditions of the previous proposition are satisfied under the following simple conditions.

Corollary 4.3.6. *Suppose that the cost function factorizes as in Assumption 4.3.4. Assume furthermore that g is unbounded, convex and with minimum at zero, and the diffusion process (3.2) has no drift (which in our framework is equivalent to $S(x) = x$, we say that such a process is in* natural scale*). Then,*

(a) *the function $\rho(a)$ is defined as the unique solution of the equation $h(a) = h(b)$, with $a \leq 0 \leq b$, and there exists an equilibrium point,*

(b) *if the function $h(R(a, \rho(a)))$ is strictly decreasing for $a \leq 0$, the equilibrium is unique.*

Proof. Take $a_0 = 0$. We have that $\pi_1(b, R(a, b)) = \pi_2(a, R(a, b))$ is equivalent to the equality $g(b) = g(a)$, thus from the fact that g is convex with a minimum at zero, the restriction of g to

$x < 0$ is an invertible function, denote it by $g_{|(-\infty,0)}$, and we can define

$$\rho(a) = \left(g_{|(-\infty,0)}\right)^{-1}(a).$$

We conclude part (a) from the fact $\rho(0) = 0$ and condition (C_1) and is fulfilled. Condition (C_2) is verified, the first two statements follow from the monotonicity of h and $a \to g(a, R(a))$ because the inequalites can be rewritten as:

$$(g(t) - g(a_2))h(R(a_2, \rho(a_2))) < (g(t) - g(a_1))h(R(a_1, \rho(a_1))) \qquad \forall (a_2, a_1, t) \text{ s.t, } a_2 < a_1 < t \leq 0,$$

$$(g(t) - g(\rho(a_2)))h(R((a_2), \rho(a_2))) < (g(t) - g(a_1))h(R(a_1, \rho(a_1))) \qquad \forall (a_2, a_1, t) \text{ s.t. } \rho(a_2) > \rho(a_1) > t \geq 0.$$

The third integral (4.6) condition in (C_2) is automatic, as $(a_0, \rho(a_0)) = (0, 0)$. □

4.4 Examples

We present below several examples where the equations of Theorem 4.3.3 can be expressed more explicitly and solved numerically. To help the presentation, for each example, we plot in an (a, b) plane the implicit curves defined by these equations. To this end, we write equation (i) in Theorem 4.3.3 as

$$F(a, b) = \pi_1(a, R(a, b)) - \pi_2(b, R(a, b)) = 0,$$

and draw first the set of its solutions. We then draw the set determined by condition (ii). Note that there are cases where there is an intersection of both curves outside the set $\{a < b\}$, these points are of no interest for our problem. In all examples the function affecting the expectation of the market is $f(x) = x$.

4.4.1 Examples with multiplicative cost

The cost function now has the form

$$c(x, y) = \max(-\lambda x, x)(1 + |y|^\beta), \quad \lambda > 0, \quad \beta \geq 1, \tag{4.8}$$

and $q_d\lambda = q_u$.

Remark 4.4.1. *In this scenario the value* $\max(-\lambda x, x)$ *could represent the maintenance cost of certain property done by a third party. This third party will change the price of its services depending on the demand of the market.*

We consider a mean reverting process $X = \{X_t\}$ that follows the stochastic differential equation

$$dX_t = -\theta X_t dt + \sigma(X_t)dW_t, \tag{4.9}$$

such that σ is a function that satisfies the conditions of Section 4.2 and $q_d\theta < 1$. Under these conditions the function $c(x, y)$ is under Assumptions 4.2.1. First observe that if we take $x^y = 0$ for all $y \in \mathbb{R}$, then $c(x, y) \geq c(x^y, y) = 0$, Second, by taking $K_y = 0$, $\alpha_y = \lambda \wedge 1$ for all $y \in \mathbb{R}$, condition (4.1) is satisfied. Finally observe that for every $y \in \mathbb{R}$ the maps $\pi_1(x, y)$, $\pi_2(x, y)$ are decreasing on x in $(-\infty, 0)$, increasing on x in $(0, \infty)$ and both conditions (i) and (ii) in Assumptions 4.2.1 are satisfied.

In the particular case when σ is constant, we can compute

$$R(a, b) = \sqrt{\frac{\sigma^2}{\theta\pi}} \left(\frac{e^{-a^2\frac{\theta}{\sigma^2}} - e^{-b^2\frac{\theta}{\sigma^2}}}{\operatorname{erf}\left(\sqrt{\frac{\theta}{\sigma^2}}b\right) - \operatorname{erf}\left(\sqrt{\frac{\theta}{\sigma^2}}a\right)} \right),$$

where $\operatorname{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$. Using Proposition 4.3.5, existence of equilibrium points holds. Furthermore, if σ is even then uniqueness also holds. In the graphical examples below σ is constant.

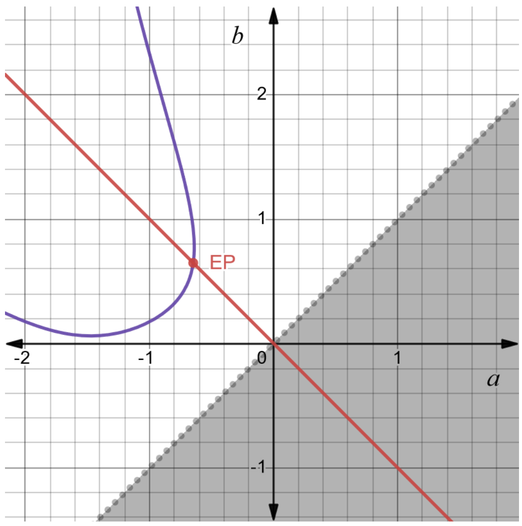


Figure 4.1: Mean reverting process (4.9) with multiplicative cost and parameters $\theta = 0.4, q_d = 0.1, \lambda = 1, \sigma = 2, \beta = 1$. The equilibrium point (EP) is $(-0.646, 0.646)$ with value 0.617.

We proceed with the calculations. Proposition 4.3.5 is used with $a_0 = 0$. We assume $\lambda \geq 1$ (in other cases symmetrical arguments can be used), $c(x,y) = \max(-\lambda x, x)(1+|y|^{\beta})$, $q := q_d = q_u \lambda^{-1}$, $\beta > 0$ and $q_d\theta < 1 \wedge \lambda^{-1}$ so the function c is a cost function.

The equality $\pi_1(b, R(a,b)) = \pi_2(a, R(a,b))$ reads as $-\lambda a = b$ taking into account that $a < 0 < b$. Furthermore if σ is an even function we deduce $R(a, -\lambda a)$ is decreasing in a so:

$$\pi_2(t, L(a)) - \pi_2(a, L(a)) = (a-t)(1 + R(a, -\lambda a)^{\beta} - q_d \lambda \theta)$$

which decreases to $-\infty$ in a.

$$\pi_1(t, L(a)) - \pi_1(-\lambda a, L(a)) = (t - \lambda a)(1 + R(a, -\lambda a)^{\beta} - q_u \theta)$$

which decrease to $-\infty$ implying that uniqueness and existence holds.

4.4.2 “Follow the market" examples

The idea is to introduce a cost function in such a way that the player has incentives to follow the market evolution. The cost function is then

$$c(x,y) = |x-y|.$$

Brownian motion with negative drift

In this case, the driving process $X = \{X_t\}$ is

$$X_t = \mu t + W_t,$$

where $\mu < 0$. We proceed to prove that Assumption 4.2.1 is satisfied. By taking $x^y = y$ for all $y \in \mathbb{R}$, then $c(x,y) \geq c(x^y, y) = 0$, Second, by taking $K_y = |y|$, $\alpha_y = 1$ for all $y \in \mathbb{R}$ then (4.1) is satisfied. Finally observe that for every $y \in \mathbb{R}$ the maps $\pi_1(x,y)$, $\pi_2(x,y)$ are decreasing on x in $(-\infty, y)$, increasing on x in (y, ∞) and both conditions (i) and (ii) in Assumptions 4.2.1 are satisfied.
The problem can be reduced to a one variable problem. The conclusions are:

- If there is a positive constant C such that

$$C(1+e^{2\mu C})(1-e^{2\mu C})^{-1} + (q_u + q_d)\mu + \mu^{-1} = 0,$$
$$\Big(\frac{C}{e^{2\mu C}-1}\Big)\frac{2e^{2\mu C}}{\mu} + \frac{-2e^{2\mu C} + 2C\mu + 1}{2\mu^2} + q_d + q_u = 0,$$

then every point of the set $\{(a, a+C), a \in \mathbb{R}\}$ is an equilibrium point.

- Otherwise there are no equilibrium points.

We put the graphics before the calculations.

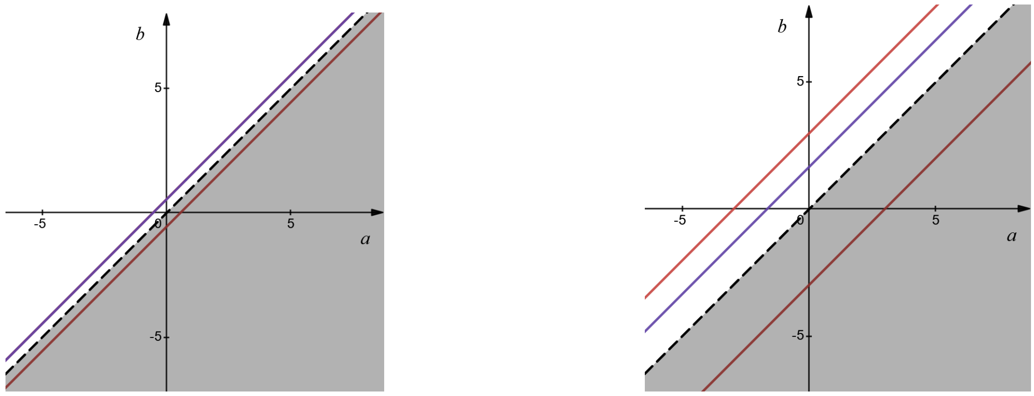


Figure 4.2: Brownian motion with drift and cost function $c(x, y) = |x - y|$. On the left ($q_u + q_d = 0.1, \mu = -0.89$) the value at equilibrium points is constant 0.848. On the right ($q_u + q_d = 2, \mu = -1$) there are no equilibrium points

In this case,

$$S'(x) = \exp(-2\mu x), \ m'(x) = 2e^{2\mu x}.$$

Therefore

$$R(a, b) = \frac{\displaystyle\int_a^b 2ue^{2\mu u} du}{\displaystyle\int_a^b 2e^{2\mu u} du} = \frac{be^{2\mu b} - ae^{2\mu a}}{e^{2\mu b} - e^{2\mu a}} - \frac{1}{2\mu}.$$

The cost function is $c(x, y) = |x - y|$. We proceed to analyze the function ρ. The notations are the same as Proposition 4.3.5. The equation $\pi_2(a, R(a, b)) = \pi_1(b, R(a, b))$ is equivalent to

$$F(a, b) = (b + a) + \frac{1}{\mu} - 2\left(\frac{be^{2\mu b} - ae^{2\mu a}}{e^{2\mu b} - e^{2\mu a}}\right) + \mu(q_u + q_d) = 0.$$

On one hand, when $a < 0$ the equation $F(a, b) = 0$ has a solution $b > 0$ because

$$a + \mu^{-1} + \mu(q_u + q_d) < 0$$

On the other, when $a \geq 0$, the equation $F(a, b) = 0$ also has a root because $b - 2R(a, b) \to -\infty$ when $b \to \infty$.

We compute the partial derivative

$$\frac{\partial F}{\partial b}(a,b) = -1\frac{(e^{2\mu(b-a)})\left(2\mu(a-b)-1+e^{2\mu(b-a)}\right)}{(1-e^{2\mu(b-a)})^2} > 0.$$

We deduce that the function ρ is well defined in all $\mathbb{R}$ and the roots of $F(a,b)$ are unique for each a. Furthermore, if C is the positive constant that satisfies the equality

$$C(1+e^{2\mu C})(1-e^{2\mu C})^{-1} = -(q_u+q_d)\mu - \mu^{-1},$$

then $F(a,a+C)=0$. So $\rho(a)=a+C$.
From Theorem 4.3.2 we know the -equilibrium points (a,b) must satisfy the equality:

$$\int_a^b (|t-R(a,b)| - b + R(a,b))2e^{2\mu t}dt + (q_u+q_d)e^{2\mu a} = 0. \tag{4.10}$$

More explicitly,

$$\begin{aligned}\int_a^b \left(\left|t+\frac{1}{2\mu}-\frac{be^{2\mu b}-ae^{2\mu a}}{e^{2\mu b}-e^{2\mu a}}\right| - b - \frac{1}{2\mu}+\frac{be^{2\mu b}-ae^{2\mu a}}{e^{2\mu b}-e^{2\mu a}}\right)2e^{2\mu t}dt&\\ +(q_u+q_d)e^{2\mu a} = 0&\end{aligned} \tag{4.11}$$

With the change of variable $u=t-b$ the equality (4.11) is equivalent to:

$$\begin{aligned}\int_{a-b}^0 \left(\left|u-\frac{(b-a)}{e^{2\mu(b-a)}-1}+\frac{1}{2\mu}\right| + \frac{(b-a)}{e^{2\mu(b-a)}-1} - \frac{1}{2\mu}\right)2e^{2\mu(u+b-a)}du\ e^{2\mu a}&\\ +(q_u+q_d)e^{2\mu a} = 0.&\end{aligned} \tag{4.12}$$

Therefore if there is a point (A,B) that satisfies (4.11) then every point (a,b) such that $b-a = B-A$ also satisfies (4.11). To solve the integral define $C := b-a$ and $K := C(\exp(2\mu C)-1)^{-1} - (2\mu)^{-1}$ so the integral in (4.12) becomes

$$\begin{aligned}\int_{-C}^0 \left(\left|u-K\right|+K\right)2e^{2\mu(C+u)}du &= 2e^{2\mu K}\int_{-C-K}^{-K}|r|e^{2\mu(C+r)}dr + Ke^{2\mu C}\frac{1-e^{-2\mu C}}{\mu}\\ &= \frac{2}{4\mu^2}e^{2\mu(K+C)}\left(e^{2\mu r}(2\mu r-1)\Big|_{-C-K}^{0} - e^{2\mu r}(2\mu r-1)\Big|_0^{-K}\right) + Ke^{2\mu C}\frac{1-e^{-2\mu C}}{\mu}\\ &= \frac{1}{2\mu^2}e^{2\mu C}(-e^{-2\mu C}(2\mu(-C-K)-1)-(-2\mu K-1)) + Ke^{2\mu C}\frac{1-e^{-2\mu C}}{\mu}\\ &= e^{2\mu C}\left(\frac{4\mu K+1}{2\mu^2}\right) + \frac{2C\mu+1}{2\mu^2} = \left(\frac{C}{e^{2\mu C}-1}\right)\frac{2e^{2\mu C}}{\mu} + \frac{-2e^{2\mu C}+2C\mu+1}{2\mu^2}.\end{aligned}$$

Solving the integral in (4.11) we conclude that a point (a, b) is an equilibrium point iff $C \coloneqq b-a$ satisfies

$$C(1 + e^{2\mu C})(1 - e^{2\mu C})^{-1} + (q_u + q_d)\mu + \mu^{-1} = 0$$
$$\left(\frac{C}{e^{2\mu C} - 1}\right)\frac{2e^{2\mu C}}{\mu} + \frac{-2e^{2\mu C} + 2C\mu + 1}{2\mu^2} + q_d + q_u = 0$$

Using 4.3.1 it can be shown that the value of the game is the same for all equilibrium points, and it is

$$-C\exp(-2\mu C)(1 - \exp(-2\mu C))^{-1} - (2\mu)^{-1} + q_d\mu$$

Ornstein Uhlenbeck process

In this case, the process $X = \{X_t\}$ follows the stochastic differential equation

$$dX_t = -\theta X_t dt + \sigma dW_t,$$

We analyze the symmetric case when $q \coloneqq q_d = q_u$ and $q\theta < 1$. In this situation, taking the same parameters as in the previous example, $c(x, y)$ is under Assumption 4.2.1. The existence of equilibrium points will hold, but uniqueness not necessarily. Essentially, the equation $\pi_1(a, R(a, b)) = \pi_2(a, R(a, b))$ is satisfied when $a = -b$ by symmetry, so similar arguments as the ones in the multiplicative case hold. However the line $a + b = 0$ is not the only set where $\pi_1(a, R(a, b)) = \pi_2(a, R(a, b))$. We show that uniqueness does not always hold, see Figure 4.3.

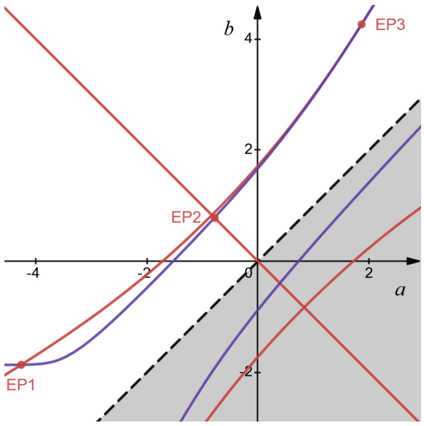


Figure 4.3: Mean reverting process with $q = 0.1$, $\theta = 3$, $\sigma = 2$, $EP1 \sim (-4.26, -1.86)$, $EP2 \sim (-0.78, 0.78)$, $EP3 \sim (1.87, 4.27)$ with the values 0.839, 0.55 and 0.84 at each equilibrium point respectively

4.5 Approximation of Nash equilibria in symmetric N-player games with mean field interaction

In this section, we present an approximation result for Nash equilibria in the N-player game corresponding to the ergodic mean field game considered above, when the number of players N tends to infinity. More precisely, we establish that an equilibrium point of the mean field game defined in (4.3.1) is an ϵ-Nash equilibrium of the corresponding N-player game (see Definition 4.5.1). These approximation results have been studied for instance in [Cao and Guo(2022)] and [Cao et al.(2023)] and the references therein. In order to formulate the approximation result, consider:

(i) A filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ that satisfies the usual conditions, where all the processes are defined.

(ii) Adapted independent Brownian motions $W, \{W^i\}_{i=1,2,\ldots}$, with the corresponding processes $X, \{X^i\}_{i=1,2,\ldots}$, each of one satisfies equation (3.2) driven by the respective $W, \{W^i\}_{i=1,2,\ldots}$.

(iii) The set of admissible controls $\mathcal{A}$ of Definition 3.2.1, that in particular assumes, given an admissible control $\eta^i = (U^i, D^i)$, the existence of the controlled process as a solution of

$$dX_t^{i,\eta^i} = \mu(X_t^{i,\eta^i})dt + \sigma(X_t^{i,\eta^i})dW_t^i + dU_t^i - dD_t^i, \quad X_0^i = x^i, \tag{4.13}$$

for each $i = 1, 2, \ldots$

For simplicity and coherence we denote by $X^{i,a,b}$ the solution to (4.13) when the i-th player chooses reflecting controls within $a < b$, denoted respectively by $U^{i,a,b}$ and $D^{i,a,b}$. As usual, we define a vector of admissible controls by

$$\Lambda = (\eta^1, \dots, \eta^N)$$

such that $\eta^i = (U^i, D^i)$ is an admissible control selected by the player i in the N-player game. Furthermore, we define

$$\Lambda^{-i} = (\eta^1, \dots, \eta^{i-1}, \eta^{i+1}, \dots, \eta^N),$$
$$(\mu, \Lambda^{-i}) = (\eta^1, \dots, \eta^{i-1}, \mu, \eta^{i+1}, \dots, \eta^N)$$

and, given a real continuous function $f(x)$, denote

$$\bar{f}_s^{-i} = \frac{1}{N-1} \sum_{j \neq i}^{N} f(X_s^{j,\eta^j}), \quad \bar{f}_s^{a,b,-i} = \frac{1}{N-1} \sum_{j \neq i}^{N} f(X_s^{j,a,b}), \tag{4.14}$$

and, given $\mu = (U, D) \in \mathcal{A}$, for (μ, Λ^{-i}), consider

$$V_N^i(\mu, \Lambda^{-i})(x) = \limsup_{T \to \infty} \frac{1}{T} \mathbf{E}_x \left(\int_0^T c\left(X_s^{i,\mu}, \bar{f}_s^{-i}\right) ds + q_u U_T^i + q_d D_T^i \right), \tag{4.15}$$

for a cost function $c(x, y)$ satisfying Assumption 4.2.1.

Definition 4.5.1. *For fixed $\epsilon > 0$ and $N \in \mathbb{N}$, a vector of admissible controls $\Lambda = (\eta^1, \dots, \eta^N)$ is called an ϵ-Nash equilibrium if for all i and all $x \in \mathbb{R}$,*

$$V_N^i(\eta^i, \Lambda^{-i})(x) \leq V_N^i(\mu, \Lambda^{-i})(x) + \epsilon, \quad \textit{for all } \mu \in \mathcal{A}.$$

We are ready to prove that the equilibrium points of the mean field game are ϵ-Nash equilibriums for the N-player game in two different situations: (i) with reflecting controls for the players and a cost function that is convex in the second variable, (ii) with general controls in $\mathcal{A}$ for the cost function $c(x, y) = |x - y|$.

Theorem 4.5.1. *Consider a cost function $c(x, y)$ that satisfies Assumption 4.2.1, and suppose that the function $f(x)$ in Def. 4.3.1 is continuous. Assume also that one of the following conditions holds:*

(i) *For every fixed x the function $y \mapsto c(x, y)$ is convex, and the set of admissible controls for each process X^i, $i = 1, \dots, N$, is the set of reflecting controls instead of $\mathcal{A}$.*
(ii) *We have $f(x) = x$ and the cost function is $c(x, y) = |x - y|$.*

Then, if (a,b) is an equilibrium point for the mean field game driven by X, given $\epsilon > 0$, the vector of controls

$$\Lambda^{a,b} = ((U^{1,a,b}, D^{1,a,b}), \dots, (U^{N,a,b}, U^{N,a,b})), \tag{4.16}$$

is an ϵ-Nash equilibrium for the N-player game, for N large enough.

In the proof of (i) we will use the following result.

Lemma 4.5.2. *Let $c(x,y)$ be a positive continuous function such that $y \mapsto c(x,y)$ is convex for each fixed x, and (X,Y) a random vector. Then*
(a) *If X and Y are independent,*

$$\mathbf{E}c(X, \mathbf{E}Y) \leq \mathbf{E}c(X,Y). \tag{4.17}$$

(b) *In the general case, statement* (4.17) *is not true.*

Proof of Lemma 4.5.2. (a) With F_X and F_Y the respective distributions of X and Y, we have

$$\begin{aligned}\mathbf{E}c(X,Y) &= \int \left(\int c(x,y) F_Y(dy) \right) F_X(dx) \\ &\geq \int c\left(x, \int y F_Y(dy)\right) F_X(dx) = \mathbf{E}c(X, \mathbf{E}Y).\end{aligned}$$

To see (b), consider $c(x,y) = |x-y|$, a standard normal random variable $X \sim \mathcal{N}(0,1)$, and the random vector $(X,Y) = (X,X)$. We have

$$\mathbf{E}c(X,Y) = \mathbf{E}|X - X| = 0 < \sqrt{\frac{2}{\pi}} = \mathbf{E}c(X, \mathbf{E}Y) = \mathbf{E}|X|,$$

giving the counter-example that concludes the proof. □

Proof of (i) *in Theorem 4.5.1.* Define the function

$$V \colon \mathcal{A} \times \{(a,b) \colon a < b\} \to \mathbb{R} \tag{4.18}$$

by the formula

$$V(\mu, (a,b)) = \limsup_{T\to\infty} \frac{1}{T} \mathbf{E}_x \left(\int_0^T c\big(X_s^\mu, \mathbf{E}_x(f(X_s^{a,b}))\big) ds + q_u U_T + q_d D_T \right),$$

where $\mu = (U,D)$. Take $\Lambda^{a,b}$ as in (4.16). The departing point is the inequality provided by the equilibrium definition:

$$V((U^{a,b},D^{a,b}),(a,b)) \leq V(\mu,(a,b)), \quad \text{for any } \mu \in \mathcal{A}. \tag{4.19}$$

Second, by equidistribution of the player's driving processes,

$$\mathbf{E}_x c(X_s^{\mu}, \mathbf{E}_x(f(X_s^{a,b}))) = \mathbf{E}_x c(X_s^{\mu}, \mathbf{E}_x(\bar{f}_s^{a,b,-i})).$$

Now, taking $c<d$ and $\mu=(U^{c,d},D^{c,d})$, by convexity and independence between the coordinates, we apply (i) in Lemma 4.5.2:

$$\mathbf{E}_x c(X_s^{c,d}, \mathbf{E}_x(\bar{f}_s^{a,b,-i})) \leq \mathbf{E}_x c(X_s^{c,d}, \bar{f}_s^{a,b,-i}),$$

Integrating in time, taking expectation and ergodic limits, combined with (4.19), it follows

$$V((U^{a,b},D^{a,b}),(a,b)) \leq V((U^{c,d},D^{c,d}),(a,b)) \leq V_N^i((U^{c,d},D^{c,d}),\Lambda_N^{a,b,-i}). \tag{4.20}$$

Now, as $f(x)$ is continuous the set $f([a,b])$ is a closed interval, denote it by $[m,M]$, and observe that

$$(X_s^{i,a,b}, \bar{f}_s^{a,b,-i}) \in [a,b]\times[m,M],$$

that is a product of closed intervals. Then, as $c(x,y)$ is uniformly continuous in this compact domain, given ϵ there exist δ s.t.

$$|c(X_s^{\mu}, \bar{f}_s^{a,b,-i}) - c(X_s^{\mu}, \mathbf{E}_x(f(X_s^{a,b}))| \leq \frac{\epsilon}{2},$$

whenever $|\bar{f}_s^{a,b,-i} - \mathbf{E}_x(f(X_s^{a,b}))| \leq \delta$. Now we apply Hoeffding's inequality for bounded random variables $m \leq f(X^{j,a,b}) \leq M$, obtaining,

$$\mathbf{P}\left(|f^{a,b,-i} - \mathbf{E}_x(f(X_s^{a,b}))| \geq \delta\right) \leq 2e^{-\frac{2\delta^2(N-1)}{(M-m)^2}}.$$

Finally, denoting $\|c\|_\infty = \max\{|c(x,y)|\colon a\leq x\leq b, m\leq y\leq M\}$, we have

$$\begin{aligned}
&\left|\frac{1}{T}\mathbf{E}_x\int_0^T \left(c(X_s^{i,a,b},\bar{f}_s^{a,b,-i}) - c(X_s^{i,a,b},\mathbf{E}_x(f(X_s^{a,b}))\right)\,ds\right| \\
&\qquad\leq \frac{\epsilon}{2} + \frac{2\|c\|_\infty}{T}\int_0^T \mathbf{P}_x\left(|\bar{f}_s^{a,b,-i} - \mathbf{E}_x(f(X_s^{a,b}))| \geq \delta\right)\,ds \\
&\qquad\qquad\leq \frac{\epsilon}{2} + 4\|c\|_\infty e^{-\frac{2\delta^2(N-1)}{(M-m)^2}} \leq \epsilon,
\end{aligned}$$

for N large enough. From this follows that, for these values of N,

$$\left|V((U^{a,b},D^{a,b}),(a,b)) - V_N^i((U^{a,b},D^{a,b}),\Lambda_N^{a,b,-i})\right| \leq \epsilon,$$

concluding, in view of (4.20), the proof of (i). □

Proof of (ii) *in Theorem 4.5.1.* As $f(x)=x$, we denote

$$\bar{X}_{s,N}^{a,b,-i} = \frac{1}{N-1}\sum_{j\neq i}^{N} X_s^{j,a,b}.$$

As (a,b) is an equilibrium point of the mean field game, given $\epsilon>0$, we have to prove that

$$V_N^i((U^{i,a,b},D^{i,a,b}),\Lambda^{a,b,-i}) \leq V_N^i(\eta,\Lambda^{a,b,-i})+\epsilon, \tag{4.21}$$

for any control $\eta\in\mathcal{A}$, for N large enough. Observe now that if for some N_0 we have $K_0 = V_{N_0}^i(\eta,\Lambda^{a,b,-i})<\infty$, then

$$\limsup_{T\to\infty}\frac{1}{T}\mathbf{E}_x\int_0^T \left|X_s^\eta - \bar{X}_{s,N_0}^{a,b,-i}\right| ds =: I_0 < \infty,$$
$$\limsup_{T\to\infty}\frac{1}{T}\mathbf{E}_x(q_uU_T) =: J_0<\infty,$$
$$\limsup_{T\to\infty}\frac{1}{T}\mathbf{E}_x(q_dD_T) =: K_0<\infty.$$

By adding and substracting $\bar{X}_{s,N_0}^{a,b,-i}$ and the triangular property, it follows

$$\limsup_{T\to\infty}\frac{1}{T}\mathbf{E}_x\int_0^T |X_s^\eta|\, ds \leq I_0+\max(|a|,|b|),$$

and in consequence

$$\max(V(\eta,(a,b)),V_N^i(\eta,\Lambda^{a,b,-i})) \leq I_0+J_0+K_0+2\max(|a|,|b|),$$

for all N and i. Then, in order to prove (4.21), we consider these controls. Now, as (a,b) is an equilibrium point, we have

$$
\begin{aligned}
&V_N^i((U^{i,a,b}, D^{i,a,b}), \Lambda^{a,b,-i}) - V_N^i(\eta, \Lambda^{a,b,-i}) \\
&\quad = V_N^i((U^{i,a,b}, D^{i,a,b}), \Lambda^{a,b,-i}) - V(\eta, (a,b)) + V(\eta, (a,b)) - V_N^i(\eta, \Lambda^{a,b,-i}) \\
&\quad \leq V_N^i((U^{i,a,b}, D^{i,a,b}), \Lambda^{a,b,-i}) - V((U^{a,b}, D^{a,bb}), (a,b)) \\
&\qquad + V(\eta, (a,b)) - V_N^i(\eta, \Lambda^{a,b,-i}) \\
&\quad \leq 2 \sup_{\eta \in \mathcal{A}} \left| V(\eta, (a,b)) - V_N^i(\eta, \Lambda^{a,b,-i}) \right| .
\end{aligned}
$$

By the triangular inequality, we observe that for $\eta \in \mathcal{A}$,

$$
\begin{aligned}
&\left| V(\eta, (a,b)) - V_N^i(\eta, \Lambda^{a,b,-i}) \right| \\
&\quad \leq \limsup_{T \to \infty} \frac{1}{T} \mathbf{E}_x \int_0^T \left| |X_s^\eta - \bar{X}_s^{a,b,-i}| - |X_s^\eta - \mathbf{E}_x(X_s^{a,b})| \right| ds \\
&\quad \leq \limsup_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{E}_x |\bar{X}_s^{a,b,-i} - \mathbf{E}(X_s^{a,b})| ds \leq \frac{b-a}{\sqrt{N-1}},
\end{aligned}
$$

because

$$
\mathbf{E}_x |\bar{X}_s^{a,b,-i} - \mathbf{E}_x(X_s^{a,b})| \leq \sqrt{\frac{1}{N-1} \mathbf{var}_x(X_s^{a,b})} \leq \frac{b-a}{\sqrt{N-1}},
$$

concluding the proof. □

Chapter 5

Ergodic and discounted singular control problems for Lévy processes

Abstract

In this chapter, we study a pair of two-sided, long time average singular control problems where the underlying process is a Lévy process. Specifically, in the first problem the objective is to minimize a discounted long-run cost by controlling the process through a pair of –increasing and decreasing– càdlàg processes. The way to do this, is through an adjoint Dynkin game. In the second problem, the framework is similar but the cost is ergodic. We solve this second case by using the abelian limit of the solution obtained in the first one. With these results, we conclude that the optimal controls within the class of càdlàg controls can be in fact found in the class of reflecting controls controls for both problems, and solved through some deterministic equations.

We study the problem posed in 2.7.1 and 2.7.2 when the underlying process is a Lévy process, the entirety of this chapter is based on [Mordecki and Oliú (2024)].

This chapter is organized as follows. In Section 5.3 verification results that give sufficient conditions for controls to be optimal are provided for both the ergodic and the discounted problems. It should be noted that our results allow us to work with approximations of the cost functions. In Sections 5.4 and 5.5, we show that the solution of the discounted problem is the solution of a Dynkin game in the sense that the continuation region of the Dynkin game is an interval whose extremes define a two-sided reflecting optimal control (see [Andersen et al. (2015)] and [Kruk et. al. (2008)]) for the discounted problem. This game is a particular case of the one proposed in [Stettner (1982)]. Furthermore, in Section 5.5, we prove that the abelian limit holds. This means that, when the discount rate ϵ decreases to zero, the normalized expected reward associated to the ϵ-discounted problem converges to a constant for each starting point, the latter constant being actually the value of the ergodic problem. Moreover, we prove that the optimal controls also give a convergent subsequence. Instead of using some criterion of continuity in the solutions like in [Cao et al.(2021)], we make use of our verification theorems in such a way that they relate both problems. Finally, Section 5.6 presents three examples, the first two when the driving Lévy process are Compound Poisson process with two-sided exponential jumps with and without Gaussian component, the third involving a strictly stable process with finite mean.

The explicit results in these examples are possible because the two-barrier problem for these processes can be solved explicitly (see [Kyprianou(2006)] and [Cai et al. (2009)]).

5.1 Introduction

The main idea of this chapter is to use an adjoint optimization problem to obtain an optimal reflecting control. In some cases, where the problem is one-sided, a link between optimal control and optimal stopping problems has been established (see [Karatzas (1983)], or more recently, for Lévy processes, [Sexton (2022), Noba and Yamazaki (2022)] as examples). In the case where the system is controlled by a process of bounded variation and the underlying process is an Itô-diffusion, verification theorems in the form of HJB equations that define a free boundary problem have been proposed and solved explicitly (see for example [Ferrari and Vargiolu (2017), Christensen et al. (2023), Kunwai et al. (2022)]). In these problems, the optimal control problem reduces to the question of finding two barriers such that the process is reflected when it reaches them. This approach does not seem to work in the case of Lévy processes, when the free boundary problem has an integro-differential equation that cannot be solved explicitly. In general when the underlying process is Lévy, it does not seem clear how to prove directly that the HJB has a solution with the needed regularity. With this problem in mind, we establish a link between the discounted problem and an auxiliary Dynkin game in such a manner that the solution of the Dynkin game exists and defines two thresholds that allows us to construct the optimal reflecting control for the long time optimization problem. Furthermore we prove that the abelian limit holds defining an ergodic problem similar to the ones in [Jack and Zervos (2006)] and [Alvarez(2018)]. We mention more reference where this relationship is relevant:

- [Karatzas and Wang (2003)] The two-sided control problem here is of finite horizon T. The elements of the control set $\mathcal{A}$ are adapted left-continuous increasing processes

$$\eta_t = U_t^+ - D_t^-,\ 0 \leq t \leq T.$$

The controlled process is $X_t = \eta_t + x$ (x represents the starting point). The bounded variation control problem is of the form

$$
\begin{aligned}
u(x) &= \inf_{\eta\in\mathcal{A}} \mathbf{E}(J(\eta,x)) \\
&= \mathbf{E}\left(\int_0^T H(t,X_t)dt + \int_{[0,T)} \gamma(t)dU^+(t) + \int_{[0,T)} \nu(t)dD^-(t) + G(X_T)\right). \quad (5.1)
\end{aligned}
$$

We enumerate some of the hypotheses:

- $H : [0,T]\times\Omega\times\mathbb{R}\to\mathbb{R}$ is (t,ω) progressively measurable for every x and for every $(t,\omega),\ x\to H(t,\omega,x)$ is convex and continouously differentaible.
- $\gamma,\ \nu[0,T]\times\Omega\to[0,\infty)$ are continuous and adapted.
- The terminal cost $G:\Omega\times\mathbb{R}\to\mathbb{R}$ is convex in the spatial variable and continuously differentiable almost surely.

We wanted to remark these hypotheses because, with the difference the controls are left-continuous and the running cost functions do not need to be C^1, this problem is the finite horizon version of 2.7.2 if we take the functions

$$
H(t,\omega,x) := c(x)e^{-\epsilon s},\ \gamma(t)\equiv e^{-\epsilon t}q_d,\ \nu\equiv e^{-\epsilon t}q_u.
$$

The adjoint Dynkin game is of the form (denoting $\mathcal{R}$ the set of stopping times):

$$
\begin{aligned}
v(x) = \sup_{\tau\in\mathcal{R}}\inf_{\sigma\in\mathcal{R}} &\mathbf{E}\left(\int_0^{\tau\wedge\sigma} H_x(t,x)dt + \gamma(\sigma)\mathbf{1}_{\sigma<\tau} - \nu(\tau)\mathbf{1}_{\tau<\sigma} + G'(x)\mathbf{1}_{\tau=\sigma=T}\right) \\
&= \inf_{\sigma\in\mathcal{R}}\sup_{\tau\in\mathcal{R}} M(x,\tau,\sigma) = \inf_{\sigma\in\mathcal{R}}\sup_{\tau\in\mathcal{R}} M(x,\tau,\sigma) = M(x,\tau^*,\sigma^*), \quad (5.2)
\end{aligned}
$$

where the equalities in the second line, and the existence of τ^*,σ^* are proved in the paper. We make a resume of the most important results:

a) There is an optimal control η^* for the problem defined (5.1). This control is obtained approximating the infimum and showing that the controls converge to an optimal control (here $T<\infty$ is used).
b) The stopping times (τ^*,σ^*) defined in (5.2) are

$$
\tau^* = \inf\{t\in[0,T], D^*>0\}\wedge T, \quad (5.3)
$$

$$
\sigma^* = \inf\{t\in[0,T], U^*>0\}\wedge T. \quad (5.4)
$$

c) The function u is a primitive of v.

The last two claims are proven by, informally speaking, differentiation. We remark that the arguments used in this paper are the most similar to the ones wielded by us in this chapter.

- [Boetius (2005)] The author pose a finite horizon problem (with terminal time T). The underlying process is an Itô-diffusion. The set $\mathcal{A}$ of admissible controls are real processes (U, D) (with different notation) defined the same way as Definition 3.2.1 with the restriction that $U_T - D_T$ has second moments. The controlled process, starting at time t_0, is a strong solution of the SDE:

$dX_t = b(s, X_s)ds + dU_s - dD_s + \sigma(s, X_s)^T dW_s,\ X_{t_0} = x,\ \sigma$ and W are multidimensional.

The dynamic cost function is a stochastic process $Y_t^{x,U,D}$ satisfying:

$$dY_s^{x,U,D} = -g(s, X_s, Y_s^{x,U,D}, Z_s)ds + a_s^T dC_s + Z_s dW_s,\ Z_T = h(X_T).$$

The control problem consists in determining the function u and control $(U^*, D^*) \in \mathcal{A}$ such that:

$$u(t,x) := u(x) = \operatorname*{ess\,inf}_{(U,D)\in\mathcal{A}} Y_0^{x,D,U} = Y_0^{x,D^*,U^*}.$$

The associated Dynkin game is of the form ($\mathcal{R}$ is the set of stopping times):

$$v(x) = v^+(x) := \operatorname*{ess\,inf}_{\sigma\in\mathcal{R}} \operatorname*{ess\,sup}_{\tau\in\mathcal{R}} R_0^x(\sigma,\tau) = \operatorname*{ess\,sup}_{\tau\in\mathcal{R}} \operatorname*{ess\,inf}_{\sigma\in\mathcal{R}} R_0^x(\sigma,\tau) = v^-(x).$$

Clearly, the equalities are proved in the paper. The process R_t^x depends on the partial derivates of g (we omit its explicit definition because it requires the addition of more auxiliary processes). The main result is that if there is a an optimal admissible control (σ^*, τ^*) for the control problem then the times (σ^*, τ^*) defined as the first times that the control is strictly greater than $U_{t_0}^*,\ D_{t_0}^*$ respectively form a Nash equilibrium for the Dynkin game. Moreover, the function v is the spatial derivate of u. The hearth of the proof lies again, informally speaking, differentiation, that is, by analytic arguments the author proves that

$$\begin{aligned} v^- \le v^+,\ \triangle_- u \le \triangle^+ u, \\ \triangle^+ u \le v^-,\ v^+ \le \triangle_- u. \end{aligned}$$

Here, $\triangle^- u, \triangle^+ u$ denote the upper right and lower left Dini derivates of the function u.

- [Guo ad Tomecek (2008)] A connection between three problems is stablished. The framework is quite general and by adding more restrictions (that is (5.5), (5.6)), the first statement of Theorem 5.2.1 can be deduced from this paper (assuming that the cost function is strictly concave and continuously differentiable). The set of starting point can be an interval, however to not indulge in the explanation of technical details, we assume in this brief resume that it is $\mathbb{R}$. The first problem is a singular control problem similar to 2.7.2, in this case, in this case for each departing point y, the value function is defined as:

$$\begin{aligned} V(y) &= \sup_{(U,D)\in\mathcal{A}_y} J(U,D,y) \\ &= \sup_{(U,D)\in\mathcal{A}_y} \mathbf{E}\left(\int_0^\infty \Pi(s, y+U_s^+ - D_s^+)ds - \int_{[0,\infty)} \gamma_+(s)dU_s - \int_{[0,\infty)} \gamma_-(s)dD_s\right). \end{aligned}$$

The set of controls $\mathcal{A}_y$ are defined again as the pair of càdlàg adapted increasing processes with the condition

$$U_0 = D_0 = 0. \tag{5.5}$$

The hypotheses are;

  - The uncontrolled process gives a finite reward, that is

$$\mathbf{E}\left(\int_0^\infty |\Pi(s,y)|ds\right) < \infty. \tag{5.6}$$

  - The function Π is strictly concave, continuously differentiable in the state space almost surely.
  - A similar condition as the one we will use in this chapter holds (see (5.10)), that is

$$\mathbf{E}\left(\int_0^\infty |\Pi_y(s,y)|ds\right) < \infty.$$

  - The processes γ_+, γ_- are adapted, non-negative and its sum is greater than zero almost surely.

The adjoint Dynkin game is similar to 2.8, but clearly, in our case γ_+, γ_- are constants (this restriction gives us enough tools to study the smoothnes of the Dynkin game). The authors obtain a third adjoint problem, that is, an *optimal switching control problem*. To define it, it is necessary to define the set of *switching controls*. A switching control is a pair $\alpha = (\tau_n, \kappa_n)_{n\geq 0}$ such that $\{\tau_n\}$ is a strictly increasing sequence of stopping times that strats at zero and converges almost surely to infinity. The discrete process

κ_n only takes values in in $\{0,1\}$ and $\kappa_{n+1} \neq \kappa_n$ for all n. and $\kappa_n \in \mathcal{F}_{\tau_n}$ (it represents a couple of regimes). The regime indicator function I_t starts at κ_0 and its value is κ_n for $t \in (\tau_n, \tau_{n+1}]$. The set $\mathcal{B}$ of *controls* for this problem are random functions functions $\alpha(z) = (\tau_n(z), \kappa_n(z))$, $z \in \mathbb{R}$ with adequate measurability and integrability conditions. The switching problem is of the form:

$$m^*(z,\kappa) := \sup_{\alpha \in \mathcal{B}, \kappa_0 = \kappa} m_+(z,\alpha) = \sup_{\alpha \in \mathcal{B}, \kappa_0 = \kappa} \mathbf{E}\left(\int_0^\infty \Pi_z(s,z) I_s ds - \sum_{n=0}^\infty \gamma(\tau_n, \kappa_n) \right),$$

with $\gamma(\tau,\kappa) = \gamma_+(\tau)\mathbf{1}_{\kappa=1} + \gamma_-(\tau)\mathbf{1}_{\kappa=0}$. The main results can be summed up as:

- There is a biyection between the singular controls and the switching controls. An optimal switching control gives (with the biyection), an optimal control for the singular control problem.
- Abusing the notation, denote $J(0,0,y)$ as the reward obtained by not controlling the process, the function $V(y) + J(0,0,y)$ can be expressed as an integral of z depending on $m_+^*(z,0)$.
- The derivate of V, generalizing the fact that the derivative of a singular control problem constitutes typically the value of an associated stopping problem, is the value of a Dynkin game (the game defined similarly to [Karatzas and Wang (2003)]) and its value is

  $$m_+^*(z,1) - m_+^*(z,0).$$

  The equilibrium point is obtained similarly to (5.3) but in this case $T = \infty$.

The hearth of the proofs are probabilistic in nature, if the underlying process is Lévy for example, this is translated to use the strong markov property and translation stationarity.

- [Guo and Tomecek (2009)] This paper uses the theory established in [Guo ad Tomecek (2008)] to solve explicitly a discounted singular control problem and its adjoint optimal switching problem, they also study the smooth-fit. The framework is the following:

  - There is a fixed interval $[a,b] \subset (-\infty,\infty)$ where the controls are defined and the singular problem is

    $$U(x,y) := \sup_{(U,D) \in \mathcal{A}'_y} J(x,y,U,D),$$

    with

    $$J(x,y,U,D) := \mathbf{E}\left(\int_0^\infty e^{-\rho s} H(Y_s) X_s^x ds - \int_0^\infty e^{-\rho s}(K_1 dU_s + K_0 dD_s) \right),$$

– subject to:

$$
\begin{aligned}
&Y_t := y + U_t - D_t,\ y \in [a,b],\\
&dX_t^x = \mu X_t^x dt + \sqrt{2}\sigma X_t^x dW_t,\ X_0 = x > 0,\\
&H : [a,b] \to \mathbb{R} \text{ is concave},\ K_1 + K_0 > 0,\ K_1 > 0,\ \mu < \rho.
\end{aligned}
$$

– The supremum is taken over all controls $(U,D) \in \mathcal{A}'_y$, where are controls that start at zero, defined as (5.5) with integrability conditions to not make $J(x,y,U,D)$ infinite.

The roadmap the authors follow is:

– Postulate the adjoint switching problem.
– Its solution is obtained through two HJB equations (one for each state).
– The bijection between the singular controls and the switching policies is used to define an optimal singular control. Two functions F and G are obtained explicitly, such that the optimal control is a reflection on a set depending on the values

$$
X_t^x - F(Y_t),\ X_t^x - G(Y_t).
$$

– Necessary conditions are given for the smooth fit (with counterexamples when they not hold).

- [Yang (2014)] In this work, the author investigates the existence of an optimal reflecting control for a singular discounted two-sided control problem when the underlying process is a multidimensional Brownian motion and the controls are exercised only in the last coordinate. To be more specific, the process $\mathbf{X} = \{\mathbf{X}_t\}_{t\geq 0}$ is of the form:

$$
\begin{aligned}
&dX_{it} = \mu_i dt + \sum_{j=1}^{m} \sigma_{ij} W_t^j, \quad j = 1,\dots,n-1,\\
&X_{nt} = \mu_n dt + \sum_{j=1}^{m} \sigma_{nj} W_t^j + U_t - D_t,\\
&\mathbf{X}_0 = x,
\end{aligned}
$$

where $W^1,\dots,W^m$ are independent Brownian motion processes and $m \geq n$. Clearly, the authors give adequate conditions for the process not to degenerate, furthermore the *admissible controls* (defined similarly to Definition 3.2.1) (U,D) must satisfy that the last coordinate of the controlled process remains in a compact set. The cost function, which the propuse is to minimize it, is of the form:

$$
\begin{aligned}
J(x,U,D) := \mathbf{E}\left(\int_0^\infty e^{-\rho s} h(\mathbf{X}) ds + \int_0^\infty e^{-\epsilon s}(f_1(\mathbf{X}_s) dU_s^c + f_2(\mathbf{X}_s) dD_s^c)\right) \\
+ \mathbf{E}_x\left(\sum_{0\le t\le\infty} e^{-\epsilon t}\left(\int_{X_{nt^-}}^{X_{nt^-}+\triangle U_t} f_1(X_{1t},\dots,X_{(n-1)t},X_{nt^-}+y)dy\right)\right) \\
+ \mathbf{E}_x\left(\sum_{0\le t\le\infty} e^{-\epsilon t}\left(\int_{X_{nt^-}}^{X_{nt^-}-\triangle D_t} f_1(X_{1t},\dots,X_{(n-1)t},X_{nt^-}+y)dy\right)\right).
\end{aligned}
$$

We make a brief explanation of these types of costs. To sum it up, with this cost the following important property holds: there is no difference in cost between pushing the process at time t an amount y or the limit when $h \to 0$ of the cost paid when we push the process at time t an amount $y - o(h)$ and at time $t + o(h)$ we push it y.
The adjoint Dynkin game is similar to 5.2 but now the derivate is only with respect to the last variable. The authors postulate necessary conditions so that the Dynkin game, using HJB equations, defines two curves $\overline{a}, \overline{b}$ that give an optimal reflecting control for the singular control problem. We must remark that the difference between the maps h and $y \to \int_{\overline{a}(\overline{x})}^{y} h_{x_n}(x_1,\dots,x_n,u)du$, with $\overline{a}(\overline{x})$ the projection of the curve a in the last coordinate, must satisfy a series of hypotheses that are given in the form of limits of HJB equations (again conditions are given for these hypotheses to hold).

- [Dianetti and Ferrari (2023)] A multidimensional singular control problem, where the costs of the controls are constant and equal to 1, is posed. The underlying process is a multidimensional diffusion. The only controlled coordinate is the first one, to be more specific the controlled diffusion is the only strong solution to the SDE:

$$
\begin{aligned}
dX_t^{1,x} &= (a_1 + b_1^1 X_t^{1,x})dt + \overline{\sigma}(X_t^{1,x})dW_t^1 + dU_t - dD_t,\ X_{0^-}^1 = x_1,\\
dX_t^{i,x} &= b_i(X_t^{1,x})dt + \overline{\sigma}(X_t^{i,x})dW_t^i,\ X_{0^-}^{i,x} = x_i,\ i = 2,\dots,d.
\end{aligned}
$$

Here, $W = (W^1,\dots,W^d)$ is a Brownian motion. The volatility is assumed to be constant or linear with state space $\mathbb{R}^d$ and $\mathbb{R}_+^d$ respectively. The main result, denoting u the value function, is that the optimal control is to reflect in the set (see 2.5.2)

$$
\mathcal{W} = \{x : |u_{x_1}(x)| \le 1\}.
$$

A HJB equation is used (which defines an adjoint Dynkin game). Observe that, with an abuse of notation, if we differentiate with respect to x_1, the expression

$$\mathcal{L}(u)(x) - \epsilon u(x) + c(x),$$

we obtain

$$\mathcal{L}(u_{x_1})(x) - \epsilon u_{x_1}(x) + c_{x_1}(x) + \sum_i b_{i,x_1}(x) u_{x_i}(x) + \sum_{i,j} u_{x_i x_j}(x) \overline{\sigma}_{x_1}(x) \overline{\sigma}(x).$$

Thus the adjoint Dynkin game must not only take into account c_{x_1} (due to the influence of the drift) and the underlying process cannot be the same (due to the influence of the non constant volatility). We remark that the discount is greater than a constant depending on the parameters of the diffusion (thus rendering the validity of the abelian limit unclear). Examples are given but not in numerical form.

- [Federico et. al.(2023)] Contrary to the rest of the resumes of the articles mentioned we proceed to talk about the probability space defined in the paper due to the importance of defining exactly what we mean by information here. A two dimensional discounted control problem where the first variable is controlled is posed. In a complete probability space $\{\Omega, \mathcal{F}, \mathbf{P})$, a Brownian motion $W = \{W\}_t$ is defined and its $\mathbf{P}$ augmented filtration is denoted as $\mathcal{F}^W = \{\mathcal{F}_t^W\}_{t \geq 0}$. There is an unknown trend μ independent of W that can take two possible real values $\mu_0 < \mu_1$. The uncontrolled process (with unknown drift) follows the stochastic differential equation:

$$dS_t = \mu dt + \eta dW_t,\ S_0 = x,\ \eta > 0.$$

The augmented natural filtration of the process $S = \{S_t\}_{t \geq 0}$ is denoted $\mathcal{F}^S = \{\mathcal{F}_t^S\}_{t \geq 0}$ and the *admissible controls* are adapted to that filtration and defined in the usual sense. The authors define the belief process $\Pi_t := \mathbf{P}(\mu = \mu_1 | \mathcal{F}_t^S),\ t \geq 0$ (see [Johnson and Peskir (2017)] and [Lipster and Shiryaev (2001), Section 4.2,]) and the controlled process as the solution of:

$$\begin{aligned} dX_t^{U,D} &= (\mu_1 \Pi_t + \mu_0 (1 - \Pi_t)) dt + \eta d\hat{W}_t + dU_t - dD_t,\ X_{0^-}^{U,D} = x \in \mathbb{R}, \\ d\Pi_t &= \frac{\mu_1 - \mu_0}{\eta} \Pi_t (1 - \Pi_t) d\hat{W}_t,\ \Pi_0 = \pi \in (0,1). \end{aligned}$$

Here $\hat{W} = \{\hat{W}_t\}_{t \geq 0}$ is an $\mathcal{F}^S$ Brownian motion (see [Lipster and Shiryaev (2001), Theorem

4.1]). The value of the singular control problem is defined as:

$$V(x,\pi) = \inf_{(U,D)\in\mathcal{A}} \mathbf{E}\left(\int_0^\infty e^{-\epsilon t}(c(X_t^{U,D})dt + q_u dU_t + q_d dD_t)\right).$$

Obviously, reasonable properties are assumed for c. To obtain the optimal control, the authors use an adjoint Dynkin game. Due to the nature of the process and the fact that c only depends on one variable, the Dynkin game is of the form

$$v(x,\pi) = \inf_\sigma \sup_\tau \mathbf{E}_{(x,\pi)}\left(\int_0^\infty e^{-\epsilon t}c'(X_t)dt - q_u e^{-\epsilon\tau}\mathbf{1}_{\tau\leq\sigma} + q_d e^{-\epsilon\sigma}\mathbf{1}_{\tau>\sigma}\right).$$

The nature of the process does not allows to obtain, at least not in an obvious way, enough properties of the function v to allow the determination of an optimal control. Therefore, the authors transform the measure $\mathbf{P}$ in the following way. Define the process $\Phi := \frac{\Pi_t}{1-\Pi_t}$, called likehood ratio process (see [Johnson and Peskir (2017)]) and the measure $\mathbf{Q}_T$ as the equivalent measure to $\mathbf{P}$ with Radon-Nykodim derivate:

$$\frac{\partial\mathbf{Q}_T}{\partial P} = \exp\left(-\frac{\mu_1-\mu_0}{\eta}\int_0^T \Pi_s dW_s - \frac{1}{2}\int_0^T \left(\frac{\mu_1-\mu_0}{\eta}\right)^2 \Pi_s^2 ds\right).$$

Then pass the limit $T\to\infty$ and obtain a probability measure $\mathbf{Q}$. Under this new law, a new singular control problem is posed. The underlying process simpler in nature. This allows the author to prove that there is an optimal reflecting control in the new singular control problem (again using an adjoint Dynkin game). Finally an explicit relation is proved between the two singular control problems.

5.2 Setting

For the reader's convenience we rewrite some of the properties of the Lévy processes discussed in 2.2.4.

5.2.1 Lévy processes, controls and cost functions

First, let us recall some of the concepts discussed in 2.2.4. Let $X = \{X_t\}_{t\geq 0}$ be a Lévy process with finite mean defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}_x)$ departing from $X_0 = x$. Assume that the filtration is right-continuous and complete (see [Jacod and Shiryaev (2003), Definition 1.3]). Denote by $\mathbf{E}_x$ the expected value associated to the probability measure $\mathbf{P}_x$, let $\mathbf{E} = \mathbf{E}_0$ and $\mathbf{P} = \mathbf{P}_0$. The Lévy-Khintchine formula characterizes the law of the process, stating

$$\phi(z) = \log\left(\mathbf{E}(e^{zX_1})\right), \qquad z = i\theta \in i\mathbb{R},$$

with

$$\phi(z) = \frac{\sigma^2}{2}z^2 + z\mu + \int_{\mathbb{R}} (e^{zy} - 1 - zy)\,\Pi(dy),$$

where $\mu = \mathbf{E}(X_1) \in \mathbb{R}$, $\sigma \geq 0$ and $\Pi(dy)$ is a non-negative measure (the *jump measure*) that satisfies in our case $\int_{\mathbb{R}}(y^2 \wedge |y|)\Pi(dy) < \infty$. This Lévy process, being a special semimartingale (see [Jacod and Shiryaev (2003), Chapter II, 2.29]), can be expressed as a sum of three independent processes

$$X_t = X_0 + \mu t + \sigma W_t + \int_{[0,t]\times\mathbb{R}} y\,\tilde{N}(ds, dy), \tag{5.7}$$

where $\tilde{N}(ds, dy) = N(ds, dy) - ds\Pi(dy)$ is a compensated Poisson random measure, $N(ds, dy)$ being the jump measure constructed from X and $\{W_t\}_{t\geq 0}$ is an independent Brownian motion. In the case when X has bounded variation, we denote $\{S_t^+\}_{t\geq 0}$, $\{S_t^-\}_{t\geq 0}$ the couple of independent subordinantors starting at zero such that for all $t \geq 0$

$$X_t = x + S_t^+ - S_t^-.$$

For general references on Lévy processes see [Bertoin (1996), Kyprianou(2006), Sato (1999)]. We proceed to define the set of *admissible controls*, in this case there is no stochastic differential equation due to the fact that Lévy processes have stationary increments. Although, already mentioned, we define again the definition *admissible controls* and *reflecting controls* for this particular family of processes.

Definition 5.2.1. *An* admissible control *is a pair of non-negative* $\mathbf{F}$*-adapted processes* (U, D) *such that:*

(i) *Each process* $U, D\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *is right continuous and nondecreasing almost surely.*

(ii) *For each* $t \geq 0$ *the random variables* U_t *and* D_t *have finite expectation.*

We denote by $\mathcal{A}$ *the set of admissible control.*

A controlled Lévy process by the pair $(U, D) \in \mathcal{A}$ is be defined as

$$X_t^{U,D} = X_t + U_t - D_t, \qquad X_0 = x,\ U_0 = u_0,\ D_0 = d_0. \tag{5.8}$$

For $a < b$ let $\{X_t^{a,b}\colon\ t \geq 0\}$ be a process defined on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}_x)$ that follows (5.8) where $U_t^{a,b}, -D_t^{a,b}$ are the respective reflections at a and b, called *reflecting controls*. At time zero, the reflecting controls are defined as $U_0^{a,b} = (a-x)^+$ and $D_0^{a,b} = (x-b)^+$ and from now on are denoted $u_0^{a,b}$ and $d_0^{a,b}$ respectively. Moreover $U^{a,b}$, $D^{a,b}$ satisfy

$$\int_0^\infty (X_t^{a,b} - a) dU_t^{a,b} = 0, \qquad \int_0^\infty (b - X_t^{a,b}) dD_t^{a,b} = 0. \tag{5.9}$$

There is an unique strong solution that satisfies (5.8) and is also a solution of the Skorokhod (5.9) problem (see [Andersen et al. (2015)]). We remark that for every exponential random variable $e(\epsilon)$ with parameter $\epsilon > 0$ independent of the process X and every $t > 0$ the random variables $U_{e(\epsilon)}^{a,b}$ $U_t^{a,b}$, $D_{e(\epsilon)}^{a,b}$, $D_t^{a,b}$, have finite mean as a consequence of [Andersen et al. (2015), Theorem 6.3] (more specifically see Proposition 5.3.1). When the process has bounded variation we also define

$$(U_t^{0,0}, D_t^{0,0}) = (S_t^-, S_t^+), \text{ for } t > 0, \quad (U_0^{0,0}, D_0^{0,0}) = (-\min\{x, 0\}, \max\{x, 0\})$$

as a reflecting control. Observe that formula (5.8) holds with $X_t^{U^{0,0}, D^{0,0}} = 0$, for all $t \geq 0$.

From now on q_u, q_d are positive constants, and we refer to them as lower barrier cost and upper barrier cost respectively. Also, denote $q = q_u + q_d$.

Definition 5.2.2. *A cost function is a convex function $c\colon \mathbb{R} \to \mathbb{R}_+$ such that*

(i) *reaches its minimum at zero,*

(ii) *there exist a pair of constants $M \geq 0$ and $N > 0$ that satisfy*

$$c(x) + M \geq N|x|, \qquad \textit{for all } x \in \mathbb{R},$$

(iii) *for every $\delta > 0$ there is a convex function $c_\delta \in C^2(\mathbb{R})$ with minimum at zero such that $\|c - c_\delta\|_\infty < \delta$ and for every $\epsilon > 0$, $x \in \mathbb{R}$*

$$\mathbf{E}_x \left(\int_0^\infty |c_\delta'(X_s)| e^{-\epsilon s} ds \right) < \infty. \tag{5.10}$$

Remark 5.2.1. *To verify (5.10), if for every $\delta > 0$ there is a function f_δ such that $|c_\delta'(x)| = f_\delta(|x|)$ and a constant $K_\delta > 0$ such that $f_\delta(x+y) \leq K_\delta f_\delta(x) f_\delta(y)$, according to [Sato (1999), Theorem 25.3, Lemma 25.5 and Theorem 30.10], it is enough to check*

$$\int_{|x|\geq 1} f_\delta(|x|) \Pi(dx) < \infty.$$

5.2.2 The ergodic and discounted control problems

Definition 5.2.3. *Given $x \in \mathbb{R}$ and a control $(U, D) \in \mathcal{A}$, we define the ergodic cost function*

$$J(x, U, D) = \limsup_{T\to\infty} \frac{1}{T}\mathbf{E}_x \left(\int_0^T c(X_s^{U,D})ds + q_u U_T + q_d D_T\right),$$

and the ergodic value function

$$G(x) = \inf_{(U,D)\in\mathcal{A}} J(x, U, D).$$

Definition 5.2.4. *Given $x \in \mathbb{R}$, a control $(U, D) \in \mathcal{A}$ and a fixed $\epsilon > 0$, we define the ϵ-discounted cost function*

$$J_\epsilon(x, U, D) = \mathbf{E}_x \left(\int_0^\infty e^{-\epsilon s}\left(c(X_s^{U,D})ds + q_u dU_s + q_d dD_s\right) + q_u u_0 + q_d d_0\right),$$

and the ϵ-discounted value function

$$G_\epsilon(x) = \inf_{(U,D)\in\mathcal{A}} J_\epsilon(x, U, D).$$

5.2.3 Main results

The most important results of the article are the link between the discounted problem and a Dynkin game, the optimality of reflecting controls for the discounted problem (see Theorem 5.2.1) and the abelian limits which give an optimal reflecting control for the ergodic problem (see Theorem 5.2.2).

Theorem 5.2.1. *Under the same notations as (5.2.4) there is a pair $a_\epsilon^* \leq 0 \leq b_\epsilon^*$ such that $G_\epsilon(x) = J_\epsilon(x, U^{a_\epsilon^*, b_\epsilon^*}, D^{a_\epsilon^*, b_\epsilon^*})$. Moreover if $c \in C^2(\mathbb{R})$, then $a_\epsilon^* < 0 < b_\epsilon^*$ and the function G_ϵ is a primitive of the function*

$$V_\epsilon(x) = \sup_{a\leq 0}\inf_{b\geq 0} \mathbf{E}_x \left(\int_0^{\tau(a)\wedge\sigma(b)} c'(X_s)e^{-\epsilon s}ds + q_d e^{-\epsilon\tau(a)}\mathbf{1}_{\tau(a)\leq\sigma(b)} - q_u e^{-\epsilon\sigma(b)}\mathbf{1}_{\sigma(b)<\tau(a)}\right),$$

with

$$\tau(a) = \inf\{t \geq 0\colon X_t \leq a\}, \quad \sigma(b) = \inf\{t \geq 0\colon X_t \geq b\}. \tag{5.11}$$

Furthermore $a_\epsilon^*, b_\epsilon^*$ satisfy:

$$a_\epsilon^* = \sup\{x < 0, V(x) = -q_u\},\ b_\epsilon^* = \inf\{x > 0, V(x) = q_d\}.$$

Theorem 5.2.2. *The ergodic value function of Definition 5.2.3 satisfies:*

(i) $\lim_{\epsilon\searrow 0}\epsilon G_\epsilon = G$ *uniformly in compacts.*

(ii) *The ergodic value function is constant.*

(iii) *There is a pair* $a^*\leq 0\leq b^*$ *such that* $G(x)=J(x,U^{a^*,b^*},D^{a^*,b^*})$, *for all* $x\in\mathbb{R}$.

5.3 Preliminary results

In order to solve the optimal control problems for the cost functions defined above, the usual approach, when the process is a diffusion, is to formulate a verification theorem with a Hamilton-Jacobi-Bellman (HJB) equation (see [Shreve et al. (1984), Harrison(1985)], or more recently, for two-sided problems, [Kunwai et al. (2022), Christensen et al. (2023)]). We use a similar approach for Lévy process. First we show that the problems do not degenerate.

Proposition 5.3.1. *For every* $\epsilon>0$, *the functions* G_ϵ *and* G *are finite. Moreover* G_ϵ *is upper continuous.*

Proof. For the ergodic value, finitude is deduced by taking a reflecting control and using [Andersen et al. (2015), Corollary 6.6]. For the ϵ-discounted value, from (70) in Theorem 6.3 of the same article, we have for $(U^{0,b},D^{0,b})$, $b>0$:

$$
\begin{aligned}
bD_t^{0,b}\leq b^2+x^2+2\int_{0+}^t X_{s-}^{0,b}dX_s+\frac{\sigma^2}{2}t&\\
+\sum_{s\leq t}\left(\mathbf{1}_{\Delta X_s\geq b}(2\Delta X_s b+b)+\mathbf{1}_{\Delta X_s\leq -b}(b^2-2b\Delta X_s)+\mathbf{1}_{|\Delta X_s|<b}\Delta X_s^2\right).&\quad(5.12)
\end{aligned}
$$

Therefore, using that the process

$$
t\to\int_{0+}^t X_{s-}^{0,b}d(X_s-s\mathbf{E}X_1)
$$

is a martingale (see Lemma A.2.1):

$$
\begin{aligned}
b\mathbf{E}_x(D_t^{0,b})\leq b^2++x^2+2b\mathbf{E}|X_1|+\frac{\sigma^2}{2}t&\\
+t\int_{\mathbb{R}}\left(\mathbf{1}_{y\geq b}(2yb+b)+\mathbf{1}_{y\leq -b}(b^2-2by)+\mathbf{1}_{|y|<b}y^2\right)\Pi(dy).&\quad(5.13)
\end{aligned}
$$

Thus, integrating by parts and by taking again a reflecting control we deduce the finitude. For the upper continuity, first notice

$$G_\epsilon(x) = \inf_{(U,D)\in\mathcal{A}} \mathbf{E}\left(\int_0^\infty e^{-\epsilon s}\left(c(x+X_s^{U,D})ds + q_u dU_s + q_d dD_s\right) + q_u u_0 + q_d d_0\right).$$

Therefore for this proposition we only work with the probability measure $\mathbf{P}$. Now fix $x \in \mathbb{R}$ and $r > 0$. Take $(U,D) \in \mathcal{A}$ such that

$$J(x,U,D) - r \leq G_\epsilon(x).$$

Take $y \in \mathbb{R}$ and let $(U^y, D^y) \in \mathcal{A}$ be defined as

$$D_0^y = d_0 + (y-x)^+,\ U_0^y = u_0 + (x-y)^+,\ D_t^y = D_t,\ U_t^y = U_t \text{ for all } t > 0.$$

It is clear $J(x,U,D) - J(y,U^y,D^y) = q_u(x-y)^+ + q_d(y-x)^+$. Therefore

$$\limsup_{y\to x} G_\epsilon(y) \leq J(x,U,D) \leq G_\epsilon(x) - r.$$

The proof is concluded because r is arbitrary. □

To avoid redundancy in the hypotheses of the theorems of this section we need the following proposition.

Proposition 5.3.2. (i) *Assume that X has unbounded variation. If a function $u \in C^2(\mathbb{R})$ linear outside an interval, then $\mathcal{L}u$ is continuous and*

$$\mathcal{L}u(x) = \mu u'(x) + \int_{\mathbb{R}} (u(x+y) - u(x) - yu'(x))\ \Pi(dy) + \frac{\sigma^2}{2}u''(x).$$

(ii) *On the other hand, if the process X has bounded variation and a function $u \in C^1(\mathbb{R})$ linear outside a compact set, then $\mathcal{L}u$ is continuous and*

$$\mathcal{L}u(x) = \mu u'(x) + \int_{\mathbb{R}} (u(x+y) - u(x) - yu'(x))\,\Pi(dy).$$

Proof. First we assume that the process has unbounded variation. Observe that

$$\int_{\mathbb{R}} |u(x) - u(x+y) - yu'(x)|\Pi(dy) \leq \sup_{y\in\mathbb{R}}(|u''(y)|)\int_{|y|<1} y^2\Pi(dy) + 2\sup_{y\in\mathbb{R}}(|u'(y)|)\int_{|y|\geq 1}|y|\Pi(dy) < \infty, \quad \text{for all } x \in \mathbb{R}. \quad (5.14)$$

Similarly observe that for all $t \geq 0$,

$$\sum_{0\leq s\leq t} |u(X_s) - u(X_{s^-}) - \Delta X_s u'(X_{s^-})| \\ \leq \sup_{y\in\mathbb{R}} |u'(y)| \sum_{0\leq s\leq t,\ |\Delta X_s|\geq 1} 2|\Delta X_s| + \sup_{y\in\mathbb{R}} |u''(y)| \sum_{0\leq s\leq t,\ |\Delta X_s|<1} \Delta X_s^2. \quad (5.15)$$

thus the process

$$\sum_{0\leq s\leq t} (u(X_s) - u(X_{s^-}) - \Delta X_s u'(X_{s^-})) - t\int_{\mathbb{R}} (u(x+y) - u(x) - yu'(x))\Pi(dy).$$

is a martingale for all $x \in \mathbb{R}$. Using this result, the inequality (5.14), Itô formula we deduce that we have to prove that the next expression goes to zero when $t \to 0$:

$$\frac{1}{t}\mathbf{E}\bigg(\int_0^t \mu(u'(x+X_{s^-}) - u'(x)) + \frac{\sigma^2}{2}(u''(x+X_{s^-}) - u''(x))ds \\ + \int_0^t \int_{\mathbb{R}} (u(x+X_{s^-}+y) - u(x+X_{s^-}) - yu'(x+X_{s^-}))d\Pi(y)ds \\ + \int_0^t \int_{\mathbb{R}} (-u(x+y) + u(x) + yu'(x))d\Pi(y)ds\bigg).$$

By dominated convergence we deduce that the limit is in fact zero. To prove the continuity it is enough to prove that the map

$$x \to \int_{\mathbb{R}} (u(x+y) - u(x) - yu(x))\Pi(dy) \quad (5.16)$$

is continuous. Using Taylor's remainder theorem:

$$\int_{\mathbb{R}} (u(x+y) - u(x) - yu'(x))\,\Pi(dy) = \\ \int_{\mathbb{R}} \left(\int_x^{x+y} u''(z)(z-x)dz\right)\Pi(dy) = \int_{\mathbb{R}} \left(\int_0^y u''(u+x)u\ du\right)\Pi(dy).$$

Therefore the continuity of the map (5.16) is obtained because u'' is uniformly continuous with compact support.
For the case of bounded variation the proof is similar, however instead of (5.14) and (5.15) we use the inequalities

$$\int_{\mathbb{R}} |u(x) - u(x+y) - yu'(x)|\Pi(dy) \leq 2\sup_{y\in\mathbb{R}}(|u'(y)|)\int_{\mathbb{R}} |y|\Pi(dy) < \infty,$$

$$\sum_{0\leq s\leq t}|u(X_s)-u(X_{s^-})-\Delta X_s u'(X_{s^-})|\leq \sup_{y\in\mathbb{R}}|u'(y)|\sum_{0\leq s\leq t}2|\Delta X_s|,$$

respectively for all $x\in\mathbb{R}$ and $t\geq 0$. Moreover, instead of using the classical Itô formula, one should use the formula for processes of finite variation see [Protter (2005), Theorem 31]. Finally, for the continuity have to prove that the function:

$$x\to \mu u'(x)+\int_{\mathbb{R}}\big(u(x+y)-u(y)-yu'(x)\big)\Pi(dy) \tag{5.17}$$

is continuous. Using the mean value theorem it can be seen that the function (5.17) is continuous (the integral $\int_{\mathbb{R}} y\Pi(dy)$ is well defined and finite). □

Theorem 5.3.3. *Consider $\epsilon>0,\delta>0$, c a cost function (with associated function c_δ), and a convex function u in the domain of the infinitesimal such that*

$$\mathcal{L}u(x)-\epsilon u(x)+c_\delta(x)\geq 0,\ -q_u\leq u'(x)\leq q_d,\quad \textit{for all } x\in\mathbb{R}. \tag{5.18}$$

Furthermore, assume that the function is linear outside an interval and

(i) *if the process is of unbounded variation, $u\in C^2(\mathbb{R})$.*

(ii) *if the process is of bounded variation, $u\in C^1(\mathbb{R})$.*

Then, under these conditions:

$$\liminf_{T\to\infty}\frac{\epsilon}{T}\mathbf{E}_x\left(\int_0^T u(X_s^{U,D})ds\right)\leq \liminf_{T\to\infty}\frac{1}{T}\mathbf{E}_x\left(\int_0^T c_\delta(X_s^{U,D})ds+q_uU_T+q_dD_T\right),$$

for all controlled process $X^{U,D}$ that satisfy the equality

$$\liminf_{T\to\infty}\frac{1}{T}\mathbf{E}_x\left(u\left(X_T^{U,D}\right)\right)=0. \tag{5.19}$$

Proof. Consider first case (i). Define the martingales $\{M_t^{(i)}\}_{t\geq 0}$ $(i=1,2)$ by

$$M_t^{(1)}=\sigma W_t+\int_{(0,t]\times\mathbb{R}} y\,\tilde{N}(ds,dy),$$

$$M_t^{(2)}=\int_{(0,t]\times\mathbb{R}}\left(u(X_{s^-}^{U,D}+y)-u(X_{s^-}^{U,D})-yu'(X_{s^-}^{U,D})\right)\tilde{N}(ds,dy).$$

Apply Itô formula:

$$
\begin{aligned}
& u(X_T^{U,D}) - u(x + u_0 - d_0) \\
&\quad = \int_{0^+}^{T} u'(X_{t^-}^{U,D}) dX_t^{U,D} + \frac{\sigma^2}{2} \int_{0^+}^{T} u''(X_{t^-}^{U,D})\, dt + M_T^{(2)} \\
&\qquad + \int_{(0,T]\times\mathbb{R}} \Big(u(X_{t^-}^{U,D} + y) - u(X_{t^-}^{U,D}) - y u'(X_{t^-}^{U,D}) \Big)\, ds \Pi(dy) \\
&\qquad + \sum_{0<s\le t} u(X_s^{U,D}) - u(X_{s^-}^{U,D} + \triangle X_s) - u'(X_{s^-}^{U,D})(\triangle U_s - \triangle D_s) \\
&\quad = \int_{0^+}^{T} \mathcal{L} u(X_{t^-}^{U,D})\, dt + \int_{0^+}^{T} u'(X_{t^-}^{U,D})\, dM_t^{(1)} + M_T^{(2)} \\
&\qquad + \int_{0^+}^{T} u'(X_{t^-}^{U,D})(dU_t^c - dD_t^c) \\
&\qquad + \sum_{0<s\le t} u(X_s^{U,D}) - u(X_{s^-}^{U,D} + \triangle X_s) \\
&\quad \ge \int_{0^+}^{T} (\epsilon u - c_\delta)\, (X_{t^-}^{U,D})\, dt - q_u dU_T - q_d dD_T + M_T^{(3)},
\end{aligned} \tag{5.20}
$$

where we used both conditions in (5.18), $M^{(3)}$ is a martingale and the equality $X_s^{U,D} = X_{s^-}^{U,D} + \triangle X_s + \triangle U_s - \triangle D_s$. We now take expectation to obtain

$$
\begin{aligned}
\mathbf{E}_x \Big(u(X_T^{U,D}) \Big) - u(x - d_0 + u_0) + \mathbf{E}_x \left(\int_{0^+}^{T} c_\delta(X_{t^-}^{U,D}) dt + q_u U_T + q_d D_T \right) \\
\ge \mathbf{E}_x \int_{0^+}^{T} \epsilon u(X_{t^-}^{U,D})\, dt.
\end{aligned}
$$

Replacing the integration limit by 0 instead of 0^+ and $X_t^{U,D}$ instead of $X_{t^-}^{U,D}$, dividing by T and taking $T \to \infty$ we conclude the proof for the case of unbounded variation, in view of (5.19). The case (ii) of bounded variation follows similarly applying Itô-Meyer formula for processes of finite variation (see [Protter (2005), Chapter II, Theorem 31]). □

The candidate u proposed in the following sections is linear outside an interval. This property allows to prove that the reflection controls are optimal.

Theorem 5.3.4. *Under the hypothesis of Theorem 5.3.3, if there is a pair of thresholds $a <$*

$0 < b$ *such that* u *also satisfies*

$$\begin{cases} \mathcal{L}u(x) - \epsilon u(x) + c_\delta(x) = 0, & \text{for all } x \in (a,b), \\ u(x) = u(a) + (a-x)q_u, & \text{for all } x \le a, \\ u(x) = u(b) + (x-b)q_d, & \text{for all } x \ge b, \end{cases}$$

then,

$$\limsup_{T\to\infty} \mathbf{E}_x \frac{1}{T}\left(\int_0^T c_\delta(X_s^{a,b})\, ds + q_u U_T^{a,b} + q_d D_T^{a,b} \right) = \limsup_{T\to\infty} \frac{\epsilon}{T} \mathbf{E}_x \left(\int_0^T u(X_s^{a,b})\, ds \right).$$

Proof. In (5.20), as $X_{s-}^{a,b} \in (a,b)$, we have

$$\int_{0+}^T \mathcal{L}u(X_{t-}^{a,b})\, dt = \int_{0+}^T (\epsilon u - c_\delta)(X_{t-}^{a,b})\, dt.$$

Furthermore, we have $dU_t^{a,b} = dD_t^{a,b} = 0$ when $X_t^{a,b} \in (a,b)$ and $u'(x) = -q_u$ for $x \le a$ and $u'(b) = q_d$ for $x \ge b$. So, by using $X_t^{a,b} - X_{t-}^{a,b} = 0$ in the support of $(U^{a,b})^c, (D^{a,b})^c$:

$$\int_{0+}^T u'(X_{t-}^{a,b})(d(U^{a,b})_t^c - d(D^{a,b})_t^c) = -q_u d(U^{a,b})_T^c - q_d d(D^{a,b})_T^c.$$

On the other hand $u(X_s^{a,b}) - u(X_{s-}^{a,b} + \triangle X_s) = -q_u \triangle U_s - q_d \triangle D_s$ for all $s > 0$. So we have an equality in (5.20). Taking limsup we obtain the result, because $u(X_T^{a,b})$ is a bounded quantity because $u(x)$ is continuous. $\square$

We proceed to show that in the discounted problem if a candidate is found it is the exact solution.

Theorem 5.3.5. *Let* $\epsilon > 0$ *and* $\delta > 0$*, suppose that* c *is a cost function and there is convex function* u *under the same hypothesis as Theorem 5.3.3. Then we have:*

$$u(x) \le \mathbf{E}_x \left(\int_0^\infty e^{-\epsilon s} \big(c_\delta(X_s^{U,D}) ds + q_u dU_s + q_d dD_s \big) + q_u u_0 + q_d d_0 \right)$$

for all controlled processes $X^{U,D}$ *that satisfy*

$$\mathbf{E}_x \left(\int_0^\infty e^{-\epsilon s} u(X_s^{U,D}) ds \right) < \infty. \tag{5.21}$$

Proof. Observe that we can assume $\mathbf{E}_x \left(\int_0^\infty e^{-\epsilon s} c_\delta(X_s^{U,D}) ds \right) < \infty$, otherwise the claim is trivial.

Let e_ϵ be an exponential random variable with parameter ϵ, independent of X. We now apply Itô's formula on the random interval $[0, e_\epsilon]$, that, with the same arguments as in the proof of Theorem 5.3.3, give

$$\mathbf{E}_x\left(u(X^{U,D}_{e_\epsilon}) - u(x - d_0 + u_0)\right) \geq \mathbf{E}_x\left(\int_{0+}^{e_\epsilon}\left((\epsilon u - c_\delta)(X^{U,D}_{s^-})\right)ds - q_u dU_{e_\epsilon} - q_d dD_{e_\epsilon}\right). \quad (5.22)$$

On the other hand for every stochastic process Y_s with finite variation independent of e_ϵ and $g(x)$ continuous we have

$$\begin{aligned}
\mathbf{E}_x \int_{0+}^{e_\epsilon} g(X^{U,D}_{s^-})dY_s &= \mathbf{E}_x\left(\int_{0+}^{\infty}\epsilon e^{-\epsilon u}\left(\int_{0+}^{u} g(X^{U,D}_{s^-})dY_s\right)du\right)\\
&= \mathbf{E}_x\left(\int_{0+}^{\infty}\left(\int_{s}^{\infty}\epsilon e^{-\epsilon u} g(X^{U,D}_{s^-})du\right)dY_s\right)\\
&= \mathbf{E}_x\left(\int_{0+}^{\infty} e^{-\epsilon s} g(X^{U,D}_{s^-})dY_s\right).
\end{aligned}$$

Therefore the inequality (5.22) is equivalent to:

$$0 \geq \mathbf{E}_x\left(\int_{0+}^{\infty} e^{-\epsilon s}\left(-q_u dU_s - q_d dD_s - c_\delta(X^{U,D}_{s^-})ds\right)\right) + u(x - d_0 + u_0).$$

Due to the fact that X, U, D are càdlàg, again, we rewrite the inequality as

$$0 \geq \mathbf{E}_x\left(\int_{0}^{\infty} e^{-\epsilon s}\left(-q_u dU_s - q_d dD_s - c_\delta(X^{U,D}_{s})ds\right)\right) + u(x - d_0 + u_0).$$

Thus, to finish the proof it is enough to prove

$$u(x) - q_u u_0 - q_d d_0 \leq u(x - d_0 + u_0),$$

which is equivalent to the inequality:

$$0 \leq u(x + u_0 - d_0) - u(x) + q_u u_0 + q_d d_0,$$

which is clearly true because $-q_u \leq u' \leq q_d$, concluding the proof of the Theorem. $\square$

Finally we formulate the discounted version of Theorem 5.3.4. We omit the proof because it is exactly the same as in the ergodic version except in the case where $x \in (a,b)^c$, the linearity of $u(x)$ must be used.

Theorem 5.3.6. *Under the hypothesis of Theorem 5.3.5, if there is a couple $a < b$ such that u*

also satisfies:

$$\begin{cases} \mathcal{L}u(x) - \epsilon u(x) + c_\delta(x) = 0, & \text{for all } x \in (a,b), \\ \qquad\qquad u(x) = u(a) + (a-x)q_u, & \text{for all } x \le a, \\ \qquad\qquad u(x) = u(b) + (x-b)q_d, & \text{for all } x \ge b, \end{cases}$$

Then,

$$\mathbf{E}_x \left(\int_0^\infty e^{-\epsilon s} \left(c_\delta(X_s^{a,b})ds + q_u dU_s^{a,b} + q_d D_s^{a,b} \right) + q_u u_0^{a,b} + q_d d_0^{a,b} \right) = u(x).$$

5.4 The associated Dynkin game

Reduction of control problems to optimal stopping problems is a possible solving control, an initial example being the proposal by [Karatzas (1983)]. In this example, a singular control problem that satisfies a verification theorem similar to Theorem 5.3.3 above is solved by finding the solution of an associated optimal stopping problem. In our situation, due to the nature of our two- sided problem, the associated problem turns out to be a Dynkin game. The relationship between singular control problems and Dynkin games has been proven in [Karatzas and Wang (2003)] for finite horizon problems, [Boetius (2005)] for diffusions and [Guo and Tomecek (2009)] in a more general setting. In fact as an alternative path to show the relationship between the control problems, the results of this last article could be adapted for Lévy processes if we assume further restrictions on the cost function (see [Guo ad Tomecek (2008), equation (13)]). Apart from the aformetioned articles, our main references are [Ekström and Peskir(2008), Peskir(2009)] from where we also borrow some notation.

5.4.1 Introduction and properties

In this section we work with the function c_δ appearing in Definition 5.2.2. To ease the notation we omit the δ, denoting it by c. The underlying process X is the Lévy process with finite expectation defined in (5.7), ϵ is a fixed positive real number. Given the discounted control problem with value function in Definition 5.2.4, the auxiliary problem will be a discounted Dynkin game with an integral cost, that we formulate through a three-dimensional process. Consider then $\{Z_t = r + t\}_{t\ge 0}$ and $\left\{I_t = w + \int_0^t e^{-\epsilon Z_s} c'(X_s)ds\right\}_{t\ge 0}$, and the process

$$\mathbf{X} = \{\mathbf{X}_t = (X_t, I_t, Z_t)\}_{t\ge 0}, \quad \mathbf{X}_0 = \mathbf{x} = (x, w, r).$$

The process $\mathbf{X}$ is strong Markov with respect to the original filtration $\mathbf{F} = \{\mathcal{F}_t\}$. As usual denote by $\mathbf{P_x}$ and $\mathbf{E_x}$ the probability measure and expected value respectively when the process starts at the point $\mathbf{x}$. For simplicity we denote $\mathbf{P}_{x,0,0}$ as $\mathbf{P}_x$ and $\mathbf{E}_{x,0,0}$ as $\mathbf{E}_x$. A stopping time is a measurable function $\tau\colon \Omega \to [0,\infty]$ s.t.

$$\{\omega\colon \tau(\omega) \leq t\} \in \mathcal{F}_t, \quad \text{for all } t \geq 0.$$

We denote the set of stopping times as $\Re$. In our setting, the first entry time to a Borel set is a stopping time. Consider now the real functions

$$G_1(x, w, r) = w - q_u e^{-\epsilon r},$$
$$G_2(x, w, r) = w + q_d e^{-\epsilon r},$$

and denote the payoff

$$M_{\mathbf{x}}(\tau, \sigma) = \mathbf{E_x}\left(G_1(\mathbf{X}_\tau)\mathbf{1}_{\{\tau<\sigma\}} + G_2(\mathbf{X}_\sigma)\mathbf{1}_{\{\tau\geq\sigma\}}\right).$$

Observe that, with the auxiliary function

$$Q_r(\tau, \sigma) = e^{-r\epsilon}\left(q_d\mathbf{1}_{\{\tau\geq\sigma\}}e^{-\epsilon\sigma} - q_u\mathbf{1}_{\{\tau<\sigma\}}e^{-\epsilon\tau}\right),$$

the payoff can be written as

$$M_{\mathbf{x}}(\tau, \sigma) = \mathbf{E_x}\left(I(\tau \wedge \sigma) + Q_r(\tau, \sigma)\right),$$

where Q_r depends only on the stopping times, not on the process X. As usual we denote Q_0 by Q. Again, for simplicity we denote $M_{x,0,0}$ as M_x. Define the value functions

$$V_1(\mathbf{x}) = \sup_\tau \inf_\sigma M_{\mathbf{x}}(\tau, \sigma), \quad V_2(\mathbf{x}) = \inf_\sigma \sup_\tau M_{\mathbf{x}}(\tau, \sigma).$$

In this section, hitting times of translated sets play a crucial role. If γ is defined by

$$\gamma = \inf\{t \geq 0\colon X_t \in A\},$$

then the stopping time γ_y is

$$\gamma_y = \inf\{t \geq 0\colon X_t \in A - y\}.$$

Definition 5.4.1 (ϵ-Dynkin Game)**.** *Given the process* $\boldsymbol{X}_t = \{(X_t, I_t, Z_t)\}_{t\geq 0}$*, the functions* G_1, G_2*,* $M_{\boldsymbol{x}}$*, and the payoffs* V_1, V_2 *defined above, the* ϵ*-Dynkin game is the problem consisting in finding two stopping times* (τ^*, σ^*) *s.t.*

$$M_{\boldsymbol{x}}(\tau, \sigma^*) \leq M_{\boldsymbol{x}}(\tau^*, \sigma^*) \leq M_{\boldsymbol{x}}(\tau^*, \sigma), \quad \textit{for all } \tau, \sigma \in \Re, \tag{5.23}$$

and the value function V_ϵ *s.t.*

$$V_\epsilon(\boldsymbol{x}) = V_1(\boldsymbol{x}) = V_2(\boldsymbol{x}) = M_{\boldsymbol{x}}(\tau^*, \sigma^*). \tag{5.24}$$

The stopping times (τ^*, σ^*) *in* (5.23) *constitute a* Nash equilibrium.

To ease the notation we denote $V_1(x, 0, 0), V_2(x, 0, 0)$ as $V_1(x), V_2(x)$ respectively. Next result is based on [Stettner (1982), Theorem 1] and [Ekström and Peskir(2008), Theorem 2.1], and gives useful properties of our Dynkin game.

Proposition 5.4.1. *The* ϵ*-Dynkin game of Definition 5.4.1 satisfies the following properties:*

(i) *The functions* V_1 *and* V_2 *are continuous. Furthermore, for all* $(x, w) \in \mathbb{R}^2$ *and* $r \geq 0$ *it holds*

$$V_1(\mathbf{x}) = V_2(\mathbf{x}).$$

From now on we denote $V_\epsilon = V_1 = V_2$.

(ii) *Define the sets*

$$D_1 = \{x \in \mathbb{R} \colon G_1(x, 0, 0) = V(x, 0, 0)\}, \tag{5.25}$$
$$D_2 = \{x \in \mathbb{R} \colon G_2(x, 0, 0) = V(x, 0, 0)\}. \tag{5.26}$$

The stopping times

$$\tau^* = \inf\{t \geq 0 \colon X_t \in D_1\} \tag{5.27}$$

$$\sigma^* = \inf\{t \geq 0 \colon X_t \in D_2\}, \tag{5.28}$$

constitute a Nash equilibrium.

(iii) *The following statements hold:*

$$\mathbf{E}_x\left(e^{-\epsilon(\sigma^*\wedge t)}V_\epsilon(X_{\sigma^*\wedge t})\right) \le V_\epsilon(x) - \mathbf{E}_x\int_0^{\sigma^*\wedge t} e^{-\epsilon s}c'(X_s)ds, \tag{5.29}$$

$$\mathbf{E}_x\left(e^{-\epsilon(\tau^*\wedge t)}V_\epsilon(X_{\tau^*\wedge t})\right) \ge V_\epsilon(x) - \mathbf{E}_x\int_0^{\tau^*\wedge t} e^{-\epsilon s}c'(X_s)ds, \tag{5.30}$$

$$\mathbf{E}_x\left(e^{-\epsilon(\tau^*\wedge\sigma^*\wedge t)}V_\epsilon(X_{\tau^*\wedge\sigma^*\wedge t})\right) = V_\epsilon(x) - \mathbf{E}_x\int_0^{\tau^*\wedge\sigma^*\wedge t} e^{-\epsilon s}c'(X_s)ds. \tag{5.31}$$

Proof of (i). The first statement is proven in [Stettner (1982), Theorem 1]. For the second statement we need to apply [Ekström and Peskir(2008), Theorem 2.1]. To do so, we only have to check $\mathbf{E}_{\mathbf{x}}(\sup_t |G_i(X_t)|) < \infty,\ i = 1, 2$. This is clear from the fact

$$\mathbf{E}_x\left(\int_0^\infty |c'(X_s)|e^{-\epsilon s}ds\right) < \infty.$$

Proof of (ii). Again, from [Ekström and Peskir(2008), Theorem 2.1], the stopping times

$$\tau^* = \inf\{u \ge 0\colon G_1(X_u, I_u, Z_u) = V(X_u, I_u, Z_u)\},$$
$$\sigma^* = \inf\{u \ge 0\colon G_2(X_u, I_u, Z_u) = V(X_u, I_u, Z_u)\},$$

define a Nash equilibrium. In order to prove (5.27) we verify

$$G_1(\mathbf{x}) = V_\epsilon(\mathbf{x}) \Leftrightarrow G_1(x, 0, 0) = V(x, 0, 0). \tag{5.32}$$

Observe that

$$\begin{aligned} V_\epsilon(x, w, r) &= \sup_\tau \inf_\sigma \mathbf{E}\Bigg(w + \int_0^{\tau\wedge\sigma} c'(X_s)e^{-\epsilon Z_s}ds + q_d\mathbf{1}_{\{\tau\ge\sigma\}}e^{-\epsilon Z_\sigma} \\ &\qquad\qquad - q_u\mathbf{1}_{\{\tau<\sigma\}}e^{-\epsilon Z_\tau}\Bigg) \\ &= w + e^{-r}\sup_\tau \inf_\sigma \mathbf{E}_{x,0,0}\Bigg(\int_0^{\tau\wedge\sigma} c'(X_s)e^{-\epsilon s}ds \\ &\qquad\qquad + q_d\mathbf{1}_{\{\tau\ge\sigma\}}e^{-\sigma\epsilon} - q_u\mathbf{1}_{\{\tau<\sigma\}}e^{-\epsilon\tau}\Bigg) \\ &= w + e^{-r}V_1(x, 0, 0), \end{aligned} \tag{5.33}$$

where we used the fact that, for every $A \in \mathcal{B}(\mathbb{R}^3)$

$$\mathbf{P}_{x,w,r}((X_t, I_t, Z_t) \in A) = \mathbf{P}_{x,0,0}((X_t, w + I_t, r + t) \in A), \quad \forall t \in [0, \infty),$$

Using (5.33) and

$$G_1(x, w, r) = w + e^{-r} G_1(x, 0, 0).$$

the equation (5.32) follows. The proof of statement (5.28) is analogous.

Proof of (iii). First consider (5.29). We apply [Peskir(2009), Theorem 2.1, equation (2.28)], and use (5.33):

$$V_\epsilon(x) \geq \mathbf{E}_x(V_\epsilon(X_{t\wedge\sigma^*}, I_{t\wedge\sigma^*}, Z_{t\wedge\sigma^*})) = \mathbf{E}_x\left(e^{-\epsilon(t\wedge\sigma^*)} V_\epsilon(X_{t\wedge\sigma^*}, 0, 0) + I_{t\wedge\sigma^*}\right),$$

that is (5.29). Inequality (5.30) is proved analogously, and (5.31) follows with the same arguments based on [Peskir(2009), (2.29)]. □

5.4.2 The value function

Its time to return to our one-dimensional setting. With a slight abuse of notation consider the functions

$$\begin{aligned} V_\epsilon(x) &= V_\epsilon(x, 0, 0), \\ G_1(x) &= G_1(x, 0, 0) = -q_u, \\ G_2(x) &= G_2(x, 0, 0) = q_d. \end{aligned}$$

The optimal stopping times that constitute the Nash equilibrium are then

$$\begin{aligned} \tau^* &= \inf\{t \geq 0 \colon V_\epsilon(X_t) = -q_u\}, \\ \sigma^* &= \inf\{t \geq 0 \colon V_\epsilon(X_t) = q_d\}. \end{aligned}$$

Observe also that

$$-q_u \leq V_\epsilon(x) \leq q_d.$$

We proceed to study properties of the value function V_ϵ of the ϵ-Dynkin game, that will allow us to construct a candidate for the verification theorems of Section 5.3. The next proposition is crucial to deduce the necessary properties of V_ϵ without giving an explicit solution.

Proposition 5.4.2. *The value function V_ϵ is non-decreasing.*

Proof. Take $x < y$. Observe first that, as Q does not depend on x,

$$\mathbf{E}_x\left(Q(\tau^*, \sigma^*_{y-x})\right) = \mathbf{E}_y\left(Q(\tau^*_{x-y}, \sigma^*)\right) = \mathbf{E}\left(Q(\tau^*_x, \sigma^*_y)\right).$$

Then,

$$\begin{aligned} V_\epsilon(x) - V_\epsilon(y) &\leq M_x(\tau^*, \sigma^*_{y-x}) - M_y(\tau^*_{x-y}, \sigma^*) \\ &= \mathbf{E}_x\left(I(\tau^* \wedge \sigma^*_{y-x}) + Q(\tau^*, \sigma^*_{y-x})\right) - \mathbf{E}_y\left(I(\tau^*_{x-y}, \sigma^*) + Q(\tau^*_{x-y}, \sigma^*)\right) \\ &= \mathbf{E}\left(\int_0^{\tau^*_x \wedge \sigma^*_y} e^{-\epsilon s}\big(c'(x + X_s) - c'(y + X_s)\big) ds\right) \leq 0, \end{aligned}$$

by the convexity of c, concluding the proof. $\square$

Proposition 5.4.3. *Define*

$$a^*_\epsilon = \sup\{x \colon V_\epsilon(x) = -q_u\}, \quad b^*_\epsilon = \inf\{x \colon V_\epsilon(x) = q_d\}.$$

Then,

(i) *the stopping times*

$$\begin{aligned} \tau^* &= \inf\{t \geq 0 \colon X_t \leq a^*_\epsilon\} = \inf\{t \geq 0 \colon V_\epsilon(X_t) = -q_u\}, \\ \sigma^* &= \inf\{t \geq 0 \colon X_t \geq b^*_\epsilon\} = \inf\{t \geq 0 \colon V_\epsilon(X_t) = q_d\}. \end{aligned}$$

constitute a Nash equilibrium, and the value function satisfies

$$V_\epsilon(x) = \begin{cases} -q_u, & \text{for } x \leq a^*_\epsilon, \\ q_d, & \text{for } x \geq b^*_\epsilon. \end{cases} \tag{5.34}$$

(ii) *Furthermore,* $a^*_\epsilon < 0 < b^*_\epsilon$.

Proof of (i). The claim is deduced from the fact that V_ϵ is increasing, $-q_u \leq V_\epsilon(x) \leq q_d\ \forall x \in \mathbb{R}$ and the definition of τ^* and σ^*.

Proof of (ii). The fact that $a^*_\epsilon \leq 0 \leq b^*_\epsilon$ and $a^*_\epsilon \neq b^*_\epsilon$ is deduced from the fact that $c'(x)$ is increasing, $c'(0) = 0$ and $G_1 < G_2$.

Now, assume $\mathbf{P}(\tau^* > 0) = 1$ and observe for all $\tau \in \Re$:

$$\begin{aligned} M(\tau^*, \sigma) &= \mathbf{E}\left(\int_0^{\tau^*\wedge\sigma} c'(X_s)e^{-\epsilon s}ds - q_u \mathbf{1}_{\tau<\sigma}e^{-\epsilon\tau} + q_d \mathbf{1}_{\tau\geq\sigma}e^{-\epsilon\sigma}\right) \\ &\leq \mathbf{E}\left(\int_0^{\sigma} |c'(X_s)|e^{-\epsilon s}ds + q_d \mathbf{1}_{\tau\geq\sigma}e^{-\epsilon\sigma}\right) \\ &= \mathbf{E}\left(\int_0^{\sigma} |c'(X_s)|e^{-\epsilon s}ds + q_d e^{-\epsilon\sigma}\right). \end{aligned} \tag{5.35}$$

Define $\sigma = \inf\{t \geq 0 : |c'(X_t)| - q_d\epsilon \geq 0\}$, which is strictly positive almost surely because $c'(0) = 0$ and X is right continuous. For this choice of σ, $\mathbf{E}\int_0^\sigma (|c'(X_s)| - q_d\epsilon)\, ds < 0$, so $M(\tau^*, \sigma^*) \leq M(\tau^*, \sigma) < q_d$. This implies that σ^* is not identically zero, and consequently $b_\epsilon^* > 0$. A similar argument shows that $a_\epsilon^* < 0$. □

The following proposition provides bounds for $(a_\epsilon^*, b_\epsilon^*)$. The reader should note that these bounds clearly are not optimal, their importance lying in the fact that in the next section it will be proven that they are all also bounds for the optimal barriers for a Dynkin game with parameter $\bar{\epsilon}$ such that $0 < \bar{\epsilon} \leq \epsilon$. For the next proposition, for $\ell > 0, y \in \mathbb{R}$ we define γ^ℓ as:

$$\gamma^\ell = \inf\left\{t \geq 0\colon\ |X_t - \ell| \geq \frac{\ell}{2}\right\}. \tag{5.36}$$

Proposition 5.4.4. *For ϵ small enough there is a constant L such that the optimal thresholds of the $\bar{\epsilon}$-Dynking game $(a_{\bar{\epsilon}}^*, b_{\bar{\epsilon}}^*) \subset [-L, L]$ for every $\bar{\epsilon} \leq \epsilon$.*

Proof. From Definition 5.2.2, for ℓ large enough, for all $x \geq \ell/2$ we have, $c'(x) \geq N/2$. Then for all $\bar{\epsilon} > 0$:

$$\mathbf{E}_\ell\left(\int_0^{\gamma^\ell} c'(X_s)e^{-\bar{\epsilon}s}ds\right) \geq \frac{N}{2}\mathbf{E}_\ell\left(\frac{1 - e^{-\bar{\epsilon}\gamma^\ell}}{\bar{\epsilon}}\right). \tag{5.37}$$

On the other hand as

$$\mathbf{E}_\ell(\gamma^\ell) = \mathbf{E}\left(\inf\left\{t \geq 0\colon |X_t| \geq \frac{\ell}{2}\right\}\right) \to \infty \text{ as } \ell \to \infty,$$

we find an ℓ s.t.

$$\mathbf{E}_\ell(\gamma^\ell) > \frac{2}{N}(q_u + q_d). \tag{5.38}$$

For a fixed ℓ satisfying (5.38), using dominated convergence and the fact $\mathbf{E}_\ell(\gamma^\ell) < \infty$ when X is not the null process, we have that

$$\mathbf{E}_\ell\left(\frac{1-e^{-\bar{\epsilon}\gamma^\ell}}{\bar{\epsilon}}\right) \nearrow \mathbf{E}_\ell(\gamma^\ell) \quad \text{as } \bar{\epsilon}\to 0.$$

Thus, using (5.38) we can take ϵ small enough such that for every $\bar{\epsilon}\leq\epsilon$ we have

$$\mathbf{E}_\ell\left(\int_0^{\gamma^\ell} c'(X_s)e^{-\bar{\epsilon}s}ds\right) \geq q_d+q_u. \tag{5.39}$$

We now take $L := 3\ell/2$ and prove that for every $\bar{\epsilon}\leq\epsilon$, $b^*_{\bar{\epsilon}}\leq L$. Assume, by contradiction, that $b^*_{\bar{\epsilon}}>L$, what implies $q_d>V(\ell)$ (to ease the notation V is the value of the $\bar{\epsilon}$-Dynkin game). Now, by (5.31), and using that $V(x)\geq -q_u$ we have

$$q_d > V(\ell) = \mathbf{E}_\ell\left(\int_0^{\gamma^\ell} c'(X_s)e^{-\bar{\epsilon}s}ds + e^{-\bar{\epsilon}\gamma^\ell}V(X_{\gamma^\ell})\right) \geq \mathbf{E}_\ell\left(\int_0^{\gamma^\ell} c'(X_s)e^{-\bar{\epsilon}s}ds\right) - q_u \geq q_d.$$

by (6.3), what is a contradiction. The other bound is analogous (thus L must be taken as the one with the greater absolute value), concluding the proof. □

Proposition 5.4.5. *The value function V_ϵ is Lipschitz continuous.*

Proof. In view of (5.34) we need to consider the pairs x, y s.t. $a^*_\epsilon\leq x<y\leq b^*_\epsilon$. As in the proof of Proposition 5.4.2, we have,

$$\begin{aligned}
0\leq V_\epsilon(y)-V_\epsilon(x)\leq M_y(\tau^*,\sigma^*_{x-y})-M_x(\tau^*_{y-x},\sigma^*) = \\
\mathbf{E}\left(\int_0^{\tau^*_y\wedge\sigma^*_x} e^{-\epsilon s}(c'(X_s+y)-c'(X_s+x))ds\right) \\
\leq \sup_{z\in[a^*-b^*,b^*-a^*]} |c''(z)||y-x|\ \mathbf{E}(\tau^*_y\wedge\sigma^*_x).
\end{aligned}$$

We conclude the proof using the fact that $\tau^*_y\wedge\sigma^*_x$ is bounded by the stopping time

$$\inf\{t\geq 0\colon X_t\notin(a^*_\epsilon-b^*_\epsilon, b^*_\epsilon-a^*_\epsilon)\},$$

that has finite expectation (see Corollary A.2.4). □

When the process X has unbounded variation it is possible to prove the continuous differentiability of the value function V_ϵ. This will be necessary, in this situation, to construct a candidate that verifies condition (i) of Theorem 5.3.3.

Proposition 5.4.6. *If the Lévy process X has unbounded variation, the function V_ϵ is differentiable, and*

$$V'_\epsilon(x) = \mathbf{E}\int_0^{\tau_x^*\wedge\sigma_x^*} c''(x+X_s)e^{-\epsilon s}\,ds.$$

Furthermore, V'_ϵ is continuous.

Proof. Take $x \in \mathbb{R}$ and $h > 0$. We obtain the bound

$$\begin{aligned} V_\epsilon(x+h) &\le \mathbf{E}_{x+h}\left(\int_0^{\tau^*\wedge\sigma^*_{-h}} c'(X_s)e^{-\epsilon s}\,ds + Q(\tau^*,\sigma^*_{-h})\right) \\ &= \mathbf{E}\left(\int_0^{\tau^*_{x+h}\wedge\sigma^*_x} c'(x+X_s+h)e^{-\epsilon s}\,ds + Q(\tau^*_{x+h},\sigma^*_x)\right). \end{aligned}$$

Similarly

$$\begin{aligned} V_\epsilon(x) &\ge \mathbf{E}_x\left(\int_0^{\tau_h^*\wedge\sigma^*} c'(X_s)e^{-\epsilon s}\,ds + Q(\tau^*_h,\sigma^*)\right) \\ &= \mathbf{E}\left(\int_0^{\tau^*_{x+h}\wedge\sigma^*_x} c'(x+X_s)e^{-\epsilon s}\,ds + Q(\tau^*_{x+h},\sigma^*_x)\right). \end{aligned}$$

Subtracting, and applying the mean value theorem, we get

$$\frac{V_\epsilon(x+h)-V_\epsilon(x)}{h} \le \mathbf{E}\left(\int_0^{\tau^*_{x+h}\wedge\sigma^*_x} c''(x+X_s+\theta h)e^{-\epsilon s}\,ds\right),$$

where $0 \le \theta \le 1$. Furthermore, If we assume

$$\lim_{h\to 0}\mathbf{E}|\tau^*_{x+h}\wedge\sigma^*_x - \tau^*_x\wedge\sigma^*_x| = 0, \tag{5.40}$$

then using the fact that $\mathbf{E}(\tau^*\wedge\sigma^*) < \infty$, $c''(x)$ is continuous and bounded in compacts, by dominated convergence

$$\limsup_{h\downarrow 0}\frac{V_\epsilon(x+h)-V_\epsilon(x)}{h} \le \mathbf{E}\left(\int_0^{\tau^*_x\wedge\sigma^*_x} c''(x+X_s)e^{-\epsilon s}\,ds\right).$$

To prove (5.40) We analyze when

$$\tau^*_{x+h}\wedge\sigma^*_x \downarrow \tau^*_x\wedge\sigma^*_x,\ \textit{a.s.}\ \text{ as } h\downarrow 0. \tag{5.41}$$

First observe that τ^*_{x+h} decreases when h decreases. Now, for $\alpha > 0$, denote $I_\alpha = \inf_{0\le t\le\alpha} X_t$. By the strong Markov property, we have

$$\mathbf{P}(\tau^*_{x+h} - \tau^*_x > \alpha) \le \mathbf{P}(I_\alpha > -h),$$

then

$$\lim_{h\downarrow 0} \mathbf{P}(\tau^*_{x+h} - \tau^*_x > \alpha) \leq \lim_{h\downarrow 0} \mathbf{P}(I_\alpha > -h) = \mathbf{P}(I_\alpha = 0).$$

We deduce that (5.41) holds when the random variable I_α has no atoms, i.e. if and only if 0 is regular for $(-\infty, 0)$. This is the situation when the process has unbounded variation (see [Alili and Kyprianou(2005)), Proposition 7]). We conclude that (5.40) holds using (5.41) and the fact that, for h small enough, the times in the sequence are dominated by the first exit time of the set $(a^* - b^*, b^* - a^*)$.

Similar arguments apply in the other 3 cases: when $h > 0$ to obtain lower bounds, and the other two situations for $h < 0$. The continuity follows taking limits under dominated convergence, with similar arguments as above. □

5.5 Construction of the candidate

In this section we first give some premilinary results and then provide the proofs of Theorems 5.2.1 and 5.2.2.

5.5.1 Properties of the primitive of the value function

Using the harmonic properties of the value function V_ϵ of the ϵ-Dynkin game introduced in the previous section, we study properties one of its primitives W ($W'(x) = V_\epsilon(x)$). Using Proposition 5.3.2 we deduce the next remark.

Remark 5.5.1. *If W is a primitive of the value function V_ϵ of the ϵ-Dynkin game in Definition 5.4.1, then W is in the domain of the infinitesimal generator $\mathcal{L}$, and the function $x \to \mathcal{L}W(x)$ is continuous.*

Lemma 5.5.1. *Let V_ϵ be the value of the ϵ-Dynkin game of Definition 5.4.1 and W a primitive of V_ϵ. Then, the map*

$$x \to \mathcal{L}W(x) + c(x) - \epsilon W(x),$$

(i) *is constant in $(a^*_\epsilon, b^*_\epsilon)$,*

(ii) *increases in (b^*_ϵ, ∞),*

(iii) *decreases in $(-\infty, a^*_\epsilon)$.*

Proof. We begin by proving (iii). For $r > 0$, let η_r be the stopping time defined as

$$\eta_r = \inf\{t \geq 0 \colon X_t \notin (-r, r)\}.$$

For $h > 0$ we need to prove that the difference

$$\lim_{r\to 0^+} \frac{\mathbf{E}\left(e^{-\epsilon\eta_r}W(X_{\eta_r}+x+h) - W(x+h) + \int_0^{\eta_r} c(X_s+x+h)e^{-\epsilon s}ds\right)}{\mathbf{E}(\eta_r)} - \lim_{r\to 0^+} \frac{\mathbf{E}\left(e^{-\epsilon\eta_r}W(X_{\eta_r}+x) - W(x) + \int_0^{\eta_r} c(X_s+x)e^{-\epsilon s}ds\right)}{\mathbf{E}(\eta_r)}$$

is equal or smaller than zero (see [Dynkin, E. B. (1965), Chapter V, Theorem 5.2]). Fix $r > 0$ such that $x + h + r < a^*_\epsilon$ and define the differentiable function H in a small interval of x (differentiability can be proven using the dominated convergence theorem):

$$H^{\eta_r}(h) := \frac{\mathbf{E}\left(e^{-\epsilon\eta_r}W(X_{\eta_r}+x+h) - W(x+h) + \int_0^{\eta_r} c(X_s+x+h)e^{-\epsilon s}ds\right)}{\mathbf{E}(\eta_r)}.$$

Then there exists a $\theta^{\eta_r}_h \in [0,1]$ such that $H^{\eta_r}(x+h) - H^{\eta_r}(x)$ is equal to

$$\frac{h}{\mathbf{E}(\eta_r)}\left(\mathbf{E}\left(e^{-\epsilon\eta_r}V_\epsilon(X_{\eta_r}+x+\theta^{\eta_r}_h h) - V_\epsilon(x+\theta^{\eta_r}_h h) + \int_0^{\eta_r} c'(X_s+x+\theta^{\eta_r}_h h)e^{-\epsilon s}ds\right)\right).$$

From (5.29) in Lemma 5.4.1 we know that the expression is equal or smaller than zero for every $r < a^*_\epsilon - h - x$, thus concluding the proof of (iii). The proofs of (i) and (ii) follow the same line of reasoning using (5.30) and (5.31) instead of (5.29). This concludes the proof of the Lemma. □

5.5.2 Continuity properties of reflecting controls

To prove Theorems 5.2.1 and 5.2.2 we need results for the reflecting controls. In this subsection we assume $b > 0$ and $x \in [0, b]$ (implying $d^{0,b}_0 = u^{0,b}_0 = 0$). Moreover, inequality (5.13) is used when applying integration by parts.

Proposition 5.5.2. *For every positive couple b, r the following inequality holds:*

$$\left|b_0\mathbf{E}_x D^{0,b_0}_t - b\mathbf{E}_x D^{0,\hat{b}}_t\right| \le 2^{-1}|b^2 - b_0^2| + t\int_{|y|<r} y^2\Pi(dy) + t|b_0 - b|\left(\mathbf{E}_x|X_1| + \sqrt{6}\int_{|y|\ge r}|y|\Pi(dy)\right),$$

for all $t > 0$.

Proof. According to [Andersen et al. (2015), Theorem 6.3], for every $b > 0$

$$2bD^{0,b}(t) = x^2 - (X^{0,b})_t^2 + 2\int_0^t X_{s-}^{0,b} dX_s + [X,X]^c(t) + J^b(t), \tag{5.42}$$

where J is an increasing and finite process defined by:

$$J^b(t) = \sum_{0<s\le t} \varphi(X_{s-}^{0,b}, \Delta X_s, b),$$

with

$$\varphi(x,y,b) = \begin{cases} -(x^2+2xy), & \text{if } y \le -x, \\ y^2, & \text{if } -x < y < b-x, \\ 2y(b-x) - (b-x)^2, & \text{if } y \ge b-x, \end{cases}$$

whose domain is the set $\{(x,y,b),\ 0 \le x \le b,\ y \in \mathbb{R}\}$. Moreover $0 \le \varphi^b(x,y) \le y^2$ for all $(x,y) \in [0,b] \times \mathbb{R}$. Furthermore, due to the fact $|X_t^{0,b_0} - X_t^{0,b}| \le |b_0 - b|$ for all $t \ge 0$ (see [Kruk et. al. (2008), Theorem 2.1]) and using (5.42) we get:

$$\begin{aligned} \left| b_0 \mathbf{E}_x D_t^{0,b_0} - b\mathbf{E}_x D_t^{0,\hat{b}} \right| &\le 2^{-1}|b^2 - b_0^2| + t\int_{|y|<r} y^2 \Pi(dy) + t|b_0 - b|\mathbf{E}_x|X_1| \\ &+ 2^{-1}\left| \mathbf{E}_x \left(\sum_{0<s\le t,\ |y|\ge r} \left(\varphi(X_{s-}^{0,b_0}, \Delta X_s, b_0) - \varphi(X_{s-}^{0,b}, \Delta X_s, b) \right) \right) \right|, \end{aligned} \tag{5.43}$$

for all $t > 0$. Therefore to finish the proof we need to prove

$$\begin{aligned} 2^{-1}\left| \mathbf{E}_x \left(\sum_{0<s\le t,\ |y|\ge r} \left(\varphi(X_{s-}^{0,b_0}, \Delta X_s, b_0) - \varphi(X_{s-}^{0,b}, \Delta X_s, b) \right) \right) \right| \\ \le t|b_0 - b|\sqrt{6}\int_{|y|\ge r} |y|\Pi(dy). \end{aligned} \tag{5.44}$$

For that endeavor, observe that φ is continuously differentiable in its domain and

$$||\nabla\varphi(x,y,b)|| \le 2\sqrt{3}|y|.$$

From this inequality and the fact

$$\|(X_{s-}^{0,b_0}, \Delta X_s, b_0) - (X_{s-}^{0,b}, \Delta X_s, b)\| \le \sqrt{2}|b_0 - b|,$$

we deduce

$$2^{-1}\left|\mathbf{E}_x\left(\sum_{0<s\le t,\ |y|\ge r}\left(\varphi(X_{s^-}^{0,b_0},\Delta X_s,b_0)-\varphi(X_{s^-}^{0,b},\Delta X_s,b)\right)\right)\right| \\ \le \sqrt{6}|b_0-b|\mathbf{E}_x\left(\sum_{0<s\le t,\ |y|\ge r}|\Delta X_s|\right).$$

Rewriting the expectation in the right term of the inequality we get

$$2^{-1}\left|\mathbf{E}_x\left(\sum_{0<s\le t,\ |y|\ge r}\left(\varphi(X_{s^-}^{0,b_0},\Delta X_s,b_0)-\varphi(X_{s^-}^{0,b},\Delta X_s,b)\right)\right)\right| \\ \le \sqrt{6}t|b_0-b|\int_{|y|\ge r}|y|\Pi(dy),$$

proving (5.44) and concluding the proposition. □

Lemma 5.5.3. *For every* $b_0>0$

(i) $\lim_{b\to b_0}\mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s}dD_s^{0,b}\right)=\mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s}dD_s^{0,b_0}\right)$,

(ii) $\lim_{b\to b_0}\lim_{T\to\infty}\frac{1}{T}\mathbf{E}_x(D_T^{0,b})=\lim_{T\to\infty}\frac{1}{T}\mathbf{E}_x(D_T^{0,b_0})$.

Proof. By integration by parts and using (5.13) we deduce that (i) is equivalent to prove that the equality

$$\lim_{b\to b_0}\mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s}D_s^{0,b}ds\right)=\mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s}D_s^{0,b_0}ds\right),$$

holds. Again this is clear from Proposition 5.5.2. The second statement is clearly deduced from the same proposition. □

Lemma 5.5.4. *If the process X has unbounded variation then*

(i) $\lim_{b\to 0}\mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s}dD_s^{0,b}\right)=\infty$,

(ii) $\lim_{b\to 0}\lim_{T\to\infty}\frac{1}{T}\mathbf{E}_x(D_T^{0,b})=\infty$.

Proof. In the case where $\sigma\neq 0$ the first item is deduced by integration by parts, [Andersen et al. (2015), Theorem 6.3] and the fact that $D_t^{0,b}$ increases in t. In that case, the second item follows from Corollary 6.6 of the same article. We proceed to study the case $\sigma=0$

and $\int_{(0,\infty)} y\Pi(dy) = \infty$. It is clear that for every $t > 0$, $\Delta D_t^{0,b} \geq (\Delta X_t - b)\mathbf{1}_{\Delta X_t \geq b}$, so

$$\mathbf{E}_x(D_T^{0,b}) \geq T \int_{y>b} (y-b)\Pi(dy).$$

Therefore, the first statement is proven by integration by parts and taking limit $b \to 0$. The second one by diving by T and then taking limit $b \to 0$ in the inequality. For the case $\sigma = 0$ and $\int_{(-\infty,0)} y\Pi(dy) = \infty$ we use a symmetrical argument as the previous case to prove

$$\lim_{T\to\infty} \frac{1}{T}\mathbf{E}_x(U_T^{0,b}) = \infty, \qquad \lim_{b\to 0} \mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s} dU_s^{0,b}\right) = \infty.$$

Then we use the fact that $X_T^{0,b} = X_t - D_t^{a,b} + U_t^{a,b} \in [0,b]$ for all T and conclude the claim. □

For the rest of the subsection X has bounded variation.

Proposition 5.5.5. *If the process $\{X_t\}$ has bounded variation and non-positive drift then for all $b > 0$*

$$|\mathbf{E}_x D_t^{0,b} - \mathbf{E}_x S_t^+| \leq t \int_{0<y\leq b} y\Pi(dy) + bt\Pi([b,\infty)).$$

Proof. Fix $b > 0$. On one hand, observe that the process $D_t^{0,b}$ can be rewritten as:

$$D_t^{0,b} = \sum_{0<s\leq t} \Delta D_s^{0,b}.$$

On the other hand it is clear that if $\Delta D_s^{0,b} \neq 0$ then $\Delta X_s > 0$. Moreover, if $\Delta X_s \geq b$ then $\Delta D_s^{0,b} = \Delta X_s - b$. Thus:

$$|\mathbf{E}_x D_t^{0,b} - \mathbf{E}_x S_t^+| \leq \mathbf{E}_x\left(\sum_{0<0s\leq t} \Delta X_s \mathbf{1}_{0<\Delta X_s\leq b} + b\mathbf{1}_{\Delta X_s > b}\right) = t\int_{0<y\leq b} y\pi(dy) + bt\pi([b,\infty)).$$

□

Lemma 5.5.6. *If the process X has bounded variation then*

(i) $\lim_{b\to 0} \mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s} dD_s^{0,b}\right) = \mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s} dS_s^+\right)$,

(ii) $\lim_{b\to b_0} \lim_{T\to\infty} \frac{1}{T}\mathbf{E}_x(D_T^{0,b}) = \lim_{T\to\infty} \frac{1}{T}\mathbf{E}_x S_T^+$.

Proof. In the case that the drift is non-positive, both statements are deduced form Proposition 5.5.5 (in the first one integration by parts must be used). In the case that the drift is positive, symmetrical results can be proven for the process $U_t^{0,b}$ and the Lemma is concluded from the fact $X_t^{0,b} \in [0,b]$ for all $t > 0$. $\square$

5.5.3 Proofs of Theorems 5.2.1 and 5.2.2

Finally, we have the necessary ingredients to prove the main results of the article.

Proof of Theorem 5.2.1. If $c \in C^2(\mathbb{R})$ the result is obtained from Remark 5.5.1 and Lemma 5.5.1, taking an adequate primitive, applied to the verification Theorems 5.3.5 and 5.3.6. Observe that the restriction in the class of controls of Theorem 5.3.5 is not restriction at all, because if the process $X^{U,D}$ does not satisfy inequality (5.21), then $J(x,U,D) = \infty$ due to condition (ii) of Definition 5.2.2 and the fact G_ϵ is linear outside an interval (a similar argument can be found in [Christensen et al. (2023), Theorem 2.10]). Furthermore, we deduce,that G_ϵ is convex due to being a primitive of the value of the ϵ-Dynkin game 5.4.1.
In the case that $c \notin C^2(\mathbb{R})$ notice that there are a couple of constants $m_0 > 0,\ l_0 > 0$ such that $|c'_\delta(x)| > m_0$ for every $x \notin [-l_0, l_0]$. Therefore, Proposition 5.4.4 can be used with the same constant L for every δ, thus every pair $(a^*_{\epsilon,\delta}, b^*_{\epsilon,\delta})$ that defines the Nash Equilibrium for the ϵ-Dynkin game with associated c'_δ belongs to the set $[-L, L]$. Naturally, taking a subsquence if necessary, we define $(a^*_\epsilon, b^*_\epsilon)$ as the limit of $(a^*_{\epsilon,\delta}, b^*_{\epsilon,\delta})$ when $\delta \to 0$.
Firstly, assume $x \in (a^*_\epsilon, b^*_\epsilon)$ and for δ small enough we can assume $x \in (a^*_{\epsilon,\delta}, b^*_{\epsilon,\delta})$. On one hand, using that we previously proved the Theorem when $c \in C^2(\mathbb{R})$, we have for every admissible control (U,D), the inequality:

$$
\begin{aligned}
&\lim_{\delta\to 0}\mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s}\big(c_\delta(X_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}})ds + q_u dU_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}} + q_d dD_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}}\big)\right)\\
&\quad\le \lim_{\delta\to 0}\mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s}\big(c_\delta(X_s^{U,D})ds + q_u dU_s + q_d dD_s\big) + q_u u_0 + q_d d_0\right)\\
&\quad= \mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s}\big(c(X_s^{U,D})ds + q_u dU_s + q_d dD_s\big) + q_u u_0 + q_d d_0\right). \qquad (5.45)
\end{aligned}
$$

Again, obbserve that the inequality holds for every pair of controls (U,D), not only for the ones under the hypothesis of 5.3.5. Therefore, we need to prove

$$
\begin{aligned}
&\lim_{\delta\to 0}\mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s}\big(c(X_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}})ds + q_u dU_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}} + q_d dD_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}}\big)\right)\\
&\qquad\qquad= \mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s}\big(c(X_s^{a^*_\epsilon,b^*_\epsilon})ds + q_u dU_s^{a^*_\epsilon,b^*_\epsilon} + q_d dD_s^{a^*_\epsilon,b^*_\epsilon}\big)\right). \qquad (5.46)
\end{aligned}
$$

The equality

$$\lim_{\delta\to 0}\mathbf{E}_x\bigg(\int_0^\infty e^{-\epsilon s}c(X_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}})ds\bigg)=\mathbf{E}_x\bigg(\int_0^\infty e^{-\epsilon s}c(X_s^{a^*_\epsilon,b^*_\epsilon})ds\bigg)$$

is deduced from the fact that the processes $\{X_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}}\}$ and $\{X_s^{a^*_\epsilon,b^*_\epsilon}\}$ are bounded in $[-l,l]$ and [Kruk et. al. (2008), Theorem 2.1], which implies that for every $A<B, C<D$:

$$|X_t^{A,B}-X_t^{C,D}|\le|X_t^{A,B}-X_t^{A,D}|+|X_t^{A,D}-X_t^{C,D}|\le|B-D|+|C-A|,\quad \text{for all } t\ge 0. \tag{5.47}$$

It remains to prove that

$$\lim_{\delta\to 0}\mathbf{E}_x\bigg(\int_0^\infty e^{-\epsilon s}dU_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}}\bigg)=\mathbf{E}_x\bigg(\int_0^\infty e^{-\epsilon s}dU_s^{a^*_\epsilon,b^*_\epsilon}\bigg).$$

In the case that the process X has bounded variation, the claim is deduced from Lemma 5.5.3 and Lemma 5.5.6. In the case of unbounded variation, again we can use can use Lemma 5.5.3 if $d=\inf_\delta(b^*_{\epsilon,\delta}-a^*_{\epsilon,\delta})$ is greater than zero. Lets assume, by contradiction $d=0$. Using Lemma 5.5.4 we have a subsequence $\{\delta_n\}_n$ such that

$$\lim_{b\to 0}\mathbf{E}_x\bigg(\int_0^\infty e^{-\epsilon s}dD_s^{a^*_{\epsilon,\delta_n},b^*_{\epsilon,\delta_n}}\bigg)=\infty,\ \text{when } n\to\infty,\ \text{for all } t>0. \tag{5.48}$$

On the other hand, denoting G_ϵ^δ the ϵ discounted value function with a cost function c_δ we observe that for every $\delta,\overline{\delta}>0,\quad ||G_\epsilon^\delta-G_\epsilon^{\overline{\delta}}||_\infty\le|\delta-\overline{\delta}|$. However, due to the assumption $d=0$ and (5.48) for each $x\in\mathbb{R}$ we have a subsequence $\{\delta_n\}_n$ such that $|G_\epsilon^{\delta_n}(x)|\to\infty$ which is absurd.
Secondly, if $x>b^*_\epsilon$ the inequality (5.46) becomes

$$\begin{aligned}
&\lim_{\delta\to 0}\mathbf{E}_x\bigg(\int_0^\infty e^{-\epsilon s}\big(c_\delta(X_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}})ds+q_u dU_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}}+q_d dD_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}}\big)+q_d(b^*_{\epsilon,\delta}-x)\bigg)\\
&\quad\le\lim_{\delta\to 0}\mathbf{E}_x\bigg(\int_0^\infty e^{-\epsilon s}\big(c_\delta(X_s^{U,D})ds+q_u dU_s+q_d dD_s\big)+q_u u_0+q_d d_0\bigg)\\
&\quad=\mathbf{E}_x\bigg(\int_0^\infty e^{-\epsilon s}\big(c(X_s^{U,D})ds+q_u dU_s+q_d dD_s\big)+q_u u_0+q_d d_0\bigg),
\end{aligned}$$

and we need to prove

$$\lim_{\delta\to 0}\mathbf{E}_x\bigg(\int_0^\infty e^{-\epsilon s}\big(c_\delta(X_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}})ds+q_u dU_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}}+q_d dD_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}}\big)+q_d(b^*_{\epsilon,\delta}-x)\bigg)$$
$$=\mathbf{E}_x\bigg(\int_0^\infty e^{-\epsilon s}\big(c(X_s^{a^*_\epsilon,b^*_\epsilon})ds+q_u dU_s^{a^*_\epsilon,b^*_\epsilon}+q_d dD_s^{a^*_\epsilon,b^*_\epsilon}\big)+q_d(b^*_\epsilon-x)\bigg),$$

which is equivalent to prove

$$\lim_{\delta\to 0}\mathbf{E}_{b^*_{\epsilon,\delta}}\bigg(\int_0^\infty e^{-\epsilon s}\big(c_\delta(X_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}})ds+q_u dU_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}}+q_d dD_s^{a^*_{\epsilon,\delta},b^*_{\epsilon,\delta}}\big)\bigg)$$
$$=\mathbf{E}_{b^*_\epsilon}\bigg(\int_0^\infty e^{-\epsilon s}\big(c(X_s^{a^*_\epsilon,b^*_\epsilon})ds+q_u dU_s^{a^*_\epsilon,b^*_\epsilon}+q_d dD_s^{a^*_\epsilon,b^*_\epsilon}\big)\bigg). \quad (5.49)$$

The proof of (5.49) follows the same reasoning as the proof of the previous case with the only precaution that in this case the continuity of c must be used due to the translation of the process. This is because similar analytical properties of the controlled process and reflections in the case where the process starts at the lower barrier can be deduced if the process starts at the upper barrier. The case $x < a^*_\epsilon$ is clearly analogous.
Finally, for the case $x\in\{a^*_\epsilon,b^*_\epsilon\}$, notice that the function $J_\epsilon(x,U^{a^*_\epsilon,b^*_\epsilon},D^{a^*_\epsilon,b^*_\epsilon})$ is equal to $G_\epsilon(x)$ if $x\notin\{a^*_\epsilon,b^*_\epsilon\}$. Therefore, from Proposition 5.3.1 we have
$\limsup_{y\to x}J_\epsilon(x,U^{a^*_\epsilon,b^*_\epsilon},D^{a^*_\epsilon,b^*_\epsilon})\le G_\epsilon(x)$ for $x\in\{a^*_\epsilon,b^*_\epsilon\}$ and thus, to conclude the proof, we need to show that

$$\lim_{x\searrow b^*_\epsilon}J_\epsilon(x,U^{a^*_\epsilon,b^*_\epsilon},D^{a^*_\epsilon,b^*_\epsilon})=J_\epsilon(b^*_\epsilon,U^{a^*_\epsilon,b^*_\epsilon},D^{a^*_\epsilon,b^*_\epsilon}).$$

This claim is deduced using from the fact if $x\ge b^*$ then

$$J_\epsilon(x,U^{a^*_\epsilon,b^*_\epsilon},D^{a^*_\epsilon,b^*_\epsilon})=q_d(x-b^*_\epsilon)+J_\epsilon(b^*_\epsilon,U^{a^*_\epsilon,b^*_\epsilon},D^{a^*_\epsilon,b^*_\epsilon}).$$

□

Proof of Theorem 5.2.2. First, observe that to prove (i) is enough to only to show the pointwise convergence and (iii). This is because G_ϵ is a convex and the third claim implies it is linear outside an interval. Furthermore let us show that it is sufficient to prove (i) for twice continuously differentiable functions: for every $\delta>0,\epsilon>0$ denote $G^\delta_\epsilon(x),G^\delta(x)$ the ϵ- discounted value function and ergodic value function respectively with underlying cost c_δ. Observe

$$|\epsilon G^\delta_\epsilon(x)-\epsilon G_\epsilon(x)|\le\epsilon\delta,\ |G^\delta(x)-G(x)|\le\delta,\ \forall x\in\mathbb{R}.$$

Therefore if

$$\limsup_\epsilon|\epsilon G^\delta_\epsilon(x)-G^\delta(x)|=0\ \forall\delta>0,$$

the first item (i) will hold. Thus, to prove (i), we can assume $c \in C^2(\mathbb{R})$.
Take $r > 0$ and $(U, D) \in \mathcal{A}$ such that

$$J(x, U, D) \leq G(x) + r.$$

Using the second hypothesis of the Definition 5.2.2 we deduce there is a constant $K > 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T}\mathbf{E}_x\left(\int_0^T (|X_s^{U,D}| + |x|)ds\right) < K. \tag{5.50}$$

On the other hand, for each $\epsilon > 0$, observe G_ϵ is in the hypothesis of the verification function in the theorems 5.3.3 and 5.3.4, thus:

$$\liminf_{T\to\infty} \frac{\epsilon}{T}\mathbf{E}_x\left(\int_0^T G_\epsilon(X_s^{U,D})ds\right) \leq G(x) + r \leq \limsup_{T\to\infty} \frac{\epsilon}{T}\mathbf{E}_x\left(\int_0^T G_\epsilon(X_s^{a_\epsilon^*,b_\epsilon^*})ds\right) + r,$$

with $(a_\epsilon^*, b_\epsilon^*)$ the optimal barriers of the ϵ-Dynkin game. We deduce, using the fact r is arbitrary, it is enough to prove:

$$\liminf_{\epsilon\to 0}\left(\liminf_{T\to\infty} \frac{\epsilon}{T}\mathbf{E}_x\left(\int_0^T G_\epsilon(X_s^{U,D})ds\right) - \epsilon G_\epsilon(x)\right) \geq 0, \tag{5.51}$$

$$\limsup_{\epsilon\to 0}\left(\limsup_{T\to\infty} \frac{\epsilon}{T}\mathbf{E}_x\left(\int_0^T G_\epsilon(X_s^{a_\epsilon,b_\epsilon})ds\right) - \epsilon G_\epsilon(x)\right) = 0. \tag{5.52}$$

The limit (5.52) is clear because $X_s^{a,b}$ is bounded and the derivate of G_ϵ is bounded in $[-q_u, q_d]$.
To prove that the limit in (5.51) holds we use (5.50) :

$$\begin{aligned}
&\liminf_{\epsilon\to 0}\left(\liminf_{T\to\infty} \mathbf{E}_x\left(\int_0^T \left(\frac{\epsilon G_\epsilon(X_s^{U,D}) - \epsilon G_\epsilon(x)}{T}\right) ds\right)\right) \\
&\geq \liminf_{\epsilon\to 0}\left(\liminf_{T\to\infty} \mathbf{E}_x\epsilon\left(\int_0^T \left(\frac{(-q_d - q_u)\left(|(X_s^{U,D})| + |x|\right)}{T}\right) ds\right)\right) \\
&\geq \liminf_{\epsilon\to 0}\left(\liminf_{T\to\infty} \epsilon\left(-q_d - q_u\right) K\right) = 0.
\end{aligned}$$

We conclude that the pointwise convergence holds because r is arbitrary. The claim (ii) follows the fact $\epsilon|G_\epsilon^\delta(x) - G_\epsilon^\delta(y)| \leq (q_d + q_u)\epsilon|x - y|$.
To prove (iii) first assume $c \in C^2(\mathbb{R})$. Naturally, we define, taking a subsequence if necessary, a^* and b^* as the limit when $\epsilon \to 0$ of the sequence $\{a_\epsilon^*\}$. Using that the pointwise convergence

in (i) holds and Theorem 5.3.4, we deduce

$$\limsup_{\epsilon\to 0} J(x, U^{a^*_\epsilon,b^*_\epsilon}, D^{a^*_\epsilon,b^*_\epsilon}) = G(x).$$

Thus it is enough to prove

$$\lim_{\epsilon\to 0} J(x, U^{a^*_\epsilon,b^*_\epsilon}, D^{a^*_\epsilon,b^*_\epsilon}) = J(x, U^{a^*,b^*}, D^{a^*,b^*}). \tag{5.53}$$

Again, by taking a subsequence if necessary we can assume $d_\epsilon := (b^*_\epsilon - a^*_\epsilon)$ is a monotone sequence. From equation (5.47) and lemmas 5.5.3, 5.5.4 and 5.5.6 we conclude the claim in a similar way as the previous Theorem.
To finish the proof for the case that the cost function is not twice continuously differentiable, we use that the optimal barriers for (a^*_δ, b^*_δ) are in a compact set and use (5.47) and the results of the previous subsection. □

5.6 Examples

In this section we use the results in [Andersen et al. (2015)] to describe the ergodic cost function $J(x,(U,D))$ of Definition 5.2.3 as the solution of a two-sided free boundary control problem and then provide examples with explicit solutions. The computations of the examples in this section are possible because the *two-barrier problem* is solvable for the family of processes chosen (see Appendix A.3 for compound Poisson processes with and without gaussian part and [Kyprianou(2006)] for strictly stable processes).

5.6.1 Introduction

In [Andersen et al. (2015), Proposition 5.1], the authors prove for a Lévy process with characteristic triplet (μ,σ^2,Π) that if $\mathbf{P}_s$ is the distribution of $X^{a,b}_s$ then $\|\mathbf{P}_s(x,\cdot) - \pi^{a,b}(\cdot)\|$ converges to zero in the norm of the total variation, where

$$\pi^{a,b}[x,b] = \mathbf{P}(X_{\eta_{[x-a-b,x-a)^c}} \geq x-a) \tag{5.54}$$

is the stationary measure of the process $X^{a,b}$ and $\eta_{[x-a-b,a-a)^c}$ denotes the first entry to the set $[x-a-b,a-a)^c$. Furthermore, in Theorem 1.1 of the same article, when $\mu = \mathbf{E}\,(X(1)) < \infty$, the following relationship is established:

$$\mathbf{E}_\pi(D^{a,b}_1) = \frac{1}{b-a}\left(2\mu\mathbf{E}_\pi(X^{a,b}_1) + \sigma^2 + \int_0^{b-a} \pi^{0,b-a}(dx)\int_{-\infty}^{\infty} \varphi(x,y,b)\Pi(dy)\right),$$

with

$$\varphi(x,y,b)=\begin{cases}-(x^2+2xy), & \text{if } y\le -x,\\ y^2, & \text{if } -x<y<b-x,\\ 2y(b-x)-(b-x)^2, & \text{if } y\ge b-x.\end{cases}$$

We know by stationarity, that

$$\lim_{T\to\infty}\frac{1}{T}\mathbf{E}_x(U_T^{a,b})=\mathbf{E}_\pi(U_1^{a,b}),\quad \lim_{T\to\infty}\frac{1}{T}\mathbf{E}_x(D_T^{a,b})=\mathbf{E}_\pi(D_1^{a,b}),$$

In consequence, from (5.8), we obtain

$$\mathbf{E}_\pi(D_1^{a,b})=\mu+\mathbf{E}_\pi(U_1^{a,b})$$

giving the following result.

Lemma 5.6.1. *Under the assumptions given in the introduction, for $a<x<b$, and $d=b-a$, the ergodic cost function of Definition 5.2.3 satisfies*

$$\begin{aligned}J(x,(U^{a,b},D^{a,b}))&=\int_{[a,b]}c(u)\pi^{a,b}(du)+\mathbf{E}_\pi(q_dD_1^{a,b}+q_uU_1^{a,b})\\ &=\int_{[0,d]}c(u+a)\pi^{0,d}(du)+q\mathbf{E}_\pi(D_1^{0,d})-\mu q_u.\end{aligned}\tag{5.55}$$

In (5.55), the lower point a only appears in the integral in the cost, and the second variable $d=b-a$ is the distance within the barriers. Condition $a^*\le 0\le b^*$ reads now $0\le -a^*\le d^*$. In what respects the discounted problem, the is reduced to find a couple $a^*\le 0\le b^*$ that minimizes

$$J_\epsilon(x,a,b)=\mathbf{E}\left(\int_0^\infty e^{-\epsilon s}\left(c(X_s^{a,b})ds+q_udU_s^{a,b}+q_ddD_s^{a,b}\right)\right).$$

Notice that the process starts at zero because as seen in Section 5.4 the optimal reflecting controls do not depend on the starting point.

5.6.2 Ergodic problem with absolute value cost for Poisson Compound Process with two-sided exponential jumps

In this example, the cost function is $c(x)=|x|$. In this case c_δ can be taken as

$$c_\delta(x)=2\delta\log(1+e^{\delta^{-1}x})-x-2\delta\log 2.$$

We consider a compound Poisson process $X = \{X_t\}_{t\geq 0}$ with double-sided exponential jumps, given by

$$X_t = x + \sum_{i=1}^{N_t^{(1)}} Y_i^{(1)} - \sum_{i=1}^{N_t^{(2)}} Y_i^{(2)}, \tag{5.56}$$

where $\{N_t^{(1)}\}_{t\geq 0}$ and $\{N_t^{(2)}\}_{t\geq 0}$ are two Poisson processes with respective positive intensities λ_1, λ_2; $\{Y_i^{(1)}\}_{i\geq 1}$ and $\{Y_i^{(2)}\}_{i\geq 1}$ are two sequences of independent exponentially distributed random variables with respective positive parameters α_1, α_2. The four processes are independent. Consequently

$$\phi(z) = \lambda_1 \frac{z}{\alpha_1 - z} - \lambda_2 \frac{z}{\alpha_2 + z}.$$

For definiteness we assume $\mathbf{E}X_1 = \lambda_1/\alpha_1 - \lambda_2/\alpha_2 < 0$. Consider the Lundberg constant ρ, i.e the positive root of $\phi(z) = 0$, given by

$$\rho = \frac{\lambda_2\alpha_1 - \lambda_1\alpha_2}{\lambda_1 + \lambda_2}.$$

Observe that $0 < \rho < \alpha_1$. To compute (5.55) we obtain the stationary distribution based on (5.54) and the fact that $\{\exp(\rho X_t)\}_{t\geq 0}$ is a martingale:

$$\begin{aligned}\pi^{0,d}(dx) = {} & \frac{\rho/\lambda_1}{\alpha_1/\lambda_1 - \alpha_2 e^{-\rho d}/\lambda_2}\delta_0(dx) + \frac{(\alpha_1+\alpha_2)/(\lambda_1+\lambda_2)}{\alpha_1/\lambda_1 - \alpha_2 e^{-\rho d}/\lambda_2}\rho e^{-\rho x}dx \\ & + \frac{\rho/\lambda_2}{\alpha_1 e^{\rho d}/\lambda_1 - \alpha_2/\lambda_2}\delta_d(dx),\end{aligned}$$

where $\delta_a(dx)$ is the Dirac measure at a. We minimize (5.55) resulting from Lemma 5.6.1. First observe, for a fixed $d \geq 0$, that the function

$$a \to \int_{[0,d]} |u + a| \pi^{0,d}(du), \quad a \in [-d, 0], \tag{5.57}$$

is convex, differentiable in $(-d, 0)$ with derivate

$$a \to \int_{[0,d]} \operatorname{sign}(u + a)\pi^{0,d}(du), \tag{5.58}$$

which is continuous and increasing in $(-d, 0)$. Therefore if the limit of the expression (5.58) $a \searrow -d$ is non negative then the minimum of the function (5.57) is reached at $a = -d$. Such limit has the value $2\pi^{0,d}(\{d\})-1$ and is equal or greater than zero if and only if

$$0 \leq d, \quad d \leq \rho^{-1}\log\left(\frac{(2\rho+\alpha_2)\lambda_1}{\alpha_1\lambda_2}\right), \tag{5.59}$$

which defines an empty set in the case $2\pi^{0,0}(\{0\})-1<0$ (a sufficient condition for this inequality to hold is that $\alpha_2 \leq \alpha_1$ and $\lambda_2 \geq \lambda_1$). With a similar analysis we deduce that that the function (5.57) is minimized at $a=0$ iff $1-2\pi^{0,d}(\{0\}) \leq 0$ and that happens if and only if (assuming $\log 0 = -\infty$)

$$0 \leq d, \quad d \leq -\rho^{-1}\log\left(\frac{(\alpha_1-2\rho)^{+}\lambda_2}{\alpha_2\lambda_1}\right). \tag{5.60}$$

In the case $2\pi^{0,d}(\{d\})-1 \leq 0$ and $1-2\pi^{0,d}(\{0\}) \leq 0$ the function (5.57) reaches its minimum at

$$a = -\rho^{-1}\log\left(2(\alpha_1+\alpha_2)\right) + \rho^{-1}\log\left(e^{-\rho d}(\alpha_2+\lambda_1\alpha_2/\lambda_2)+\alpha_1+\lambda_2\alpha_1/\lambda_1\right) \tag{5.61}$$

On the other hand, from [Andersen et al. (2015), page 70], we get

$$\mathbf{E}_\pi(D_1^{0,d}) = \frac{\rho\lambda_1}{\alpha_1}\left(\frac{e^{-\rho_d}\left(1/\lambda_1+1/\lambda_2\right)}{\alpha_1/\lambda_1-\alpha_2 e^{-\rho d}/\lambda_2}\right). \tag{5.62}$$

Thus, after computing $\int_{[0,d]}|a+u|\pi^{0,d}(du)$, we deduce that the cost function to be minimized, that depends on reflecting controls at $a<0<b$, after the change $d=b-a$, is

$$\begin{aligned} J(a,d) = \Bigg(&\frac{-a\rho}{\lambda_1} + e^{-\rho d}(d+a)\frac{\rho}{\lambda_2} + \frac{q\rho\lambda_1}{\alpha_1}\left(e^{-\rho d}\left(\lambda_1^{-1}+\lambda_2^{-1}\right)\right) \\ &+ \left(\frac{\alpha_1+\alpha_2}{\lambda_1+\lambda_2}\right)\left(e^{-\rho d}\left(-a-d-\rho^{-1}\right) - a + 2\rho^{-1}e^{a\rho} - \rho^{-1}\right)\Bigg) \\ &\times \left(\frac{\alpha_1}{\lambda_1} - \frac{\alpha_2 e^{-\rho d}}{\lambda_2}\right)^{-1}. \end{aligned}$$

Upon inspection, we conclude that the candidates for the optimal barriers are $a=0, d=0$ or d minimizing the expression above with a equal to $0, -d$ or the value in (5.61). To illustrate we put numeric examples to show that the four possible cases can happen:

If $\lambda_1=1,\ \lambda_2=2, \alpha_1=2,\ \alpha_2=1,\ q=3,$ then $a^*=0,\ d^*\sim 4.005.$

If $\lambda_1=1,\ \lambda_2=2,\ \alpha_1=2, \alpha_2=1,\ q=0.1,$ then $a^*=0,\ d^*=0.$

If $\lambda_1=1,\ \lambda_2=1,\ \alpha_1=4, \alpha_2=1,\ q=5,$ then $a^*\sim -0.272,\ d^*\sim 0.966.$

The last case, giving $a^* = -d^* \neq 0$ is very sensitive to the parameter variations

$$\begin{aligned}&\text{If } \lambda_1 = 9.999985 \times 10^5,\ \lambda_2 = 1,\ \alpha_1 = 1 \times 10^6,\ \alpha_2 = 5 \times 10^{-7},\\ &q = 0.5,\ \text{then } -a^* = d^* \sim 0.0202.\end{aligned}$$

In certain sense, the case where the optimum is reached at zero and a negative barrier can be seen as pathological because the expected value is negative yet it is optimal to keep the process in the negative line.
It should be noted, due to the nature of this process and the cost value, that without loss of generality on the parameters and also the Lundberg constant can be taken as one.

5.6.3 Discounted problem with quadratic cost for a Compound Poisson process with two-sided exponential jumps and Gaussian Noise

In this case, $\epsilon > 0$ is fixed, $c(x) = x^2/2$. We assume that the Lévy process process $\{X_t\}_{t\geq 0}$ has non-zero mean defined by

$$X_t = x + \sigma W_t + \sum_{i=1}^{N_t^{(1)}} Y_i^{(1)} - \sum_{i=1}^{N_t^{(2)}} Y_i^{(2)}, \tag{5.63}$$

with $\{W_t\}_{t\geq 0}$ a Brownian motion, $\sigma > 0$, and $\{N_t^{(1)}\}_{t\geq 0}$,$\{N_t^{(2)}\}_{t\geq 0}$,$\{Y_i^{(1)}\}_{i\geq 1}$, $\{Y_i^{(2)}\}_{i\geq 1}$ as in the previous example, the five processes are independent (for more information about the first exit time of an interval for this process see [Cai et al. (2009)]). Therefore

$$\phi(z) = \frac{\sigma^2}{2} z^2 + \lambda_1 \frac{z}{\alpha_1 - z} - \lambda_2 \frac{z}{\alpha_2 + z}.$$

Now we find a pair $a^* \leq b^*$ such that

$$G_\epsilon(x) = J_\epsilon(x, (U^{a^*,b^*}, D^{a^*,b^*})).$$

Taking $\tau(a),\ \sigma(b)$ as in (5.11)

$$M_x(a,b) = \mathbf{E}_x\bigg(\int_0^{\tau(a)\wedge\sigma(b)} e^{-\epsilon s} X_s ds - q_u e^{-\epsilon\tau(a)} \mathbf{1}_{\{\tau(a)<\sigma(b)\}} + q_d e^{-\epsilon\sigma(b)} \mathbf{1}_{\{\sigma(b)<\tau(a)\}} \bigg),$$

and applying Theorem 5.2.1, to solve the discounted control problem, we need to find $a^* < 0 <$

b^* such that

$$M(a^*, b^*) = \sup_{a<0} \inf_{b>0} M(a,b), \tag{5.64}$$

where we assume $X_0 = 0$ because (a^*, b^*) do not depend on the starting point. Using [Cai et al. (2009), Theorem 3.1], we deduce:

$$\begin{aligned} &M(a,b) = \\ &(e^{-\rho_3 b}, e^{-\rho_4 b}, e^{-\rho_1 a}, e^{-\rho_2 a})\mathbf{N}_{b-a}^{-1} \begin{pmatrix} -\epsilon^{-2}(\lambda_1/\alpha_1 - \lambda_2/\alpha_2) - \epsilon^{-1}b + q_d \\ -\alpha_1^{-1}(\epsilon^{-2}(\lambda_1/\alpha_1 - \lambda_2/\alpha_2) + \epsilon^{-1}\alpha_1^{-1} + \epsilon^{-1}b - q_d) \\ -\epsilon^{-2}(\lambda_1/\alpha_1 - \lambda_2/\alpha_2) - \epsilon^{-1}a - q_u \\ -\alpha_2^{-1}(\epsilon^{-2}(\lambda_1/\alpha_1 - \lambda_2/\alpha_2) - \epsilon^{-1}\alpha_2^{-1} + \epsilon^{-1}a + q_u) \end{pmatrix} \\ &+ \frac{\lambda_1/\alpha_1 - \lambda_2/\alpha_2}{\epsilon^2}. \end{aligned}$$

With

$$\mathbf{N}_{b-a} = \begin{pmatrix} 1 & 1 & e^{-\rho_1(a-b)} & e^{-\rho_2(a-b)} \\ \dfrac{1}{\alpha_1 - \rho_3} & \dfrac{1}{\alpha_1 - \rho_4} & \dfrac{e^{-\rho_1(a-b)}}{\alpha_1 - \rho_1} & \dfrac{e^{-\rho_2(a-b)}}{\alpha_1 - \rho_2} \\ e^{\rho_3(a-b)} & e^{\rho_4(a-b)} & 1 & 1 \\ \dfrac{e^{\rho_3(a-b)}}{\alpha_2 + \rho_3} & \dfrac{e^{\rho_4(a-b)}}{\alpha_2 + \rho_4} & \dfrac{1}{\alpha_2 + \rho_1} & \dfrac{1}{\alpha_2 + \rho_2} \end{pmatrix}$$

where ρ_i $(i = 1,2,3,4)$ are the non-zero roots of the function $z \to \phi(z) - \epsilon$ that satisfy $\rho_2 < -\alpha_2 < \rho_1 < 0 < \rho_3 < \alpha_1 < \rho_4$. . This matrix is always non-singular (see [Cai et al. (2009), Proposition 3.1]).
For the parameters:

$$q_d = 1,\ q_u = 1,\ \alpha_1 = 2,\ \alpha_2 = 1,\ \lambda_2 = 1,\ \lambda_1 = 1,\ \epsilon = 1,\ \sigma = \sqrt{2},$$

the solutions of the equation $\phi(z) - \epsilon = 0$ are:

$$\rho_1 \sim -0.489,\ \rho_2 \sim -1.898,\ \rho_3 \sim 0.849,\ \rho_4 \sim 2.537,$$

and the equilibrium point is $a^* \sim -2.017, b^* \sim 2.311$.

5.6.4 Ergodic problem with quadratic cost for strictly stable processes

In this example $c(x) = x^2$, and the Lévy process X is strictly $\alpha-$stable with parameter $\alpha \in (1,2), 0 < c^+ < c^-$. In other words X is a pure jump process with finite mean and triplet

$(0, 0, \Pi)$, with jump measure

$$\Pi(dx) = \begin{cases} c^+ x^{-\alpha-1} dx, & x > 0, \\ c^- |x|^{-\alpha-1} dx, & x < 0. \end{cases}$$

The characteristic exponent is

$$\phi(i\theta) : \ = |\theta|^\alpha (c^+ + c^-) \Big(1 - i \mathrm{sgn}(\theta) \tan \big(\frac{\pi\alpha}{2} \big) \frac{c^+ - c^-}{c^+ + c^-} \Big).$$

(see [Kyprianou(2006), page 10]). The two-sided exit problem can be solved using the scaling property, that is $X_t \stackrel{d}{=} k^{\frac{-1}{\alpha}} X_{tk}$ for every $t > 0,\ k > 0$ (see [Kyprianou(2006)]). The stationary measure has Beta density $\pi^{0,d}(x)$ on $[0, d]$ with parameters $(\alpha\rho, \alpha(1-\rho))$, i.e.

$$\pi^{0,d}(x) = \frac{1}{d\beta(\alpha\rho, \alpha(1-\rho))} \left(1 - \frac{x}{d}\right)^{\alpha\rho - 1} \left(\frac{x}{d}\right)^{\alpha(1-\rho)-1},$$

where

$$\beta(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} \, dt,$$

is the Beta function and

$$\rho = \frac{1}{2} + (\pi\alpha)^{-1} \arctan \left(\left(\frac{c^+ - c^-}{c^+ + c^-} \right) \tan(\alpha\pi/2) \right).$$

From this

$$\int_0^d x\pi^{0,d}(x) \, dx = d\rho, \qquad \int_0^d (x - d\rho)^2 \pi^{0,d}(x) \, dx = d^2 \frac{\rho(1-\rho)}{\alpha + 1}, \tag{5.65}$$

and $\mathbf{E}_\pi(D_1^{0,d})$ that can be expressed explicitly (see [Andersen et al. (2015), page 114]):

$$\mathbf{E}_\pi(D_1^{0,d}) = \frac{c^- \beta(2 - \alpha\rho, \alpha\rho) + c^+ \beta(2 - \alpha(1-\rho), \alpha(1-\rho))}{\beta(\alpha\rho, \alpha(1-\rho)) \alpha(\alpha - 1)(2 - \alpha)} \frac{1}{d^{\alpha-1}} = \mathbf{E}_\pi(D_1^{0,1}) \frac{1}{d^{\alpha-1}}. \tag{5.66}$$

We solve explicitly the ergodic problem and prove that the optimal barriers (a^*, b^*) are unique. First notice that we can discard the problematic cases of trivial barriers because $\mathbf{E}(X_\infty) = \infty$ and $\mathbf{E}(D_1^{0,d}) \to \infty$ when $d \to 0$.

Observe now that for each fixed $d > 0$ the optimal lower barrier a satisfies

$$a = -\int_0^d x\pi^{0,d}(x) dx = -d\rho.$$

therefore the ergodic problem has an objective function depending only on d, given by

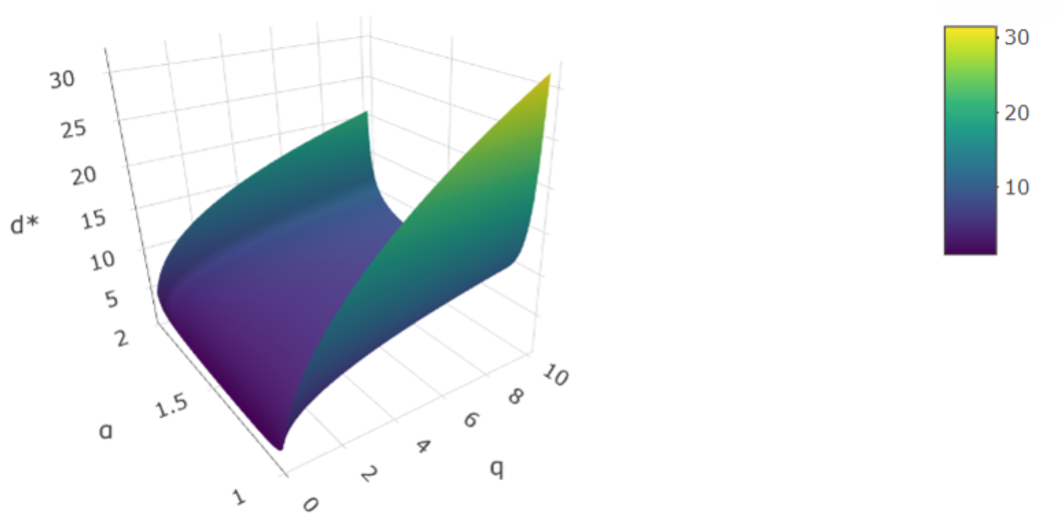
$$J(d) = \int_0^d (x - d\rho)^2 \pi^d(x)\, dx + q\mathbf{E}(D_1^{0,d})$$
$$= d^2 \frac{\rho(1-\rho)}{\alpha+1} + \frac{1}{d^{\alpha-1}} q\mathbf{E}(D_1^{0,1}). \tag{5.67}$$

By differentiation:

$$d^* = \left(\frac{(\alpha^2 - 1) q\mathbf{E}(D_1^{0,1})}{2\rho(1-\rho)} \right)^{1/(\alpha+1)}.$$

For example when $q = 1,\ c^- = 2,\ c^+ = 1, \alpha = 1.5$ the ergodic value is reached at $d^* \sim 2.850$ and $a^* = -1.230$.

We present an illustrative graphic where $c_- = 2, c^+ = 1$, the domain is $(\alpha, q) \in (1,2) \times (0,10)$ and the output is the optimal d^*:



Chapter 6

Stationary mean field games for two-sided Control Lévy problems

Abstract

In this chapter, we study a probabilistic mean field game driven by a Lévy process. The formulation is similar to Chapter 4 in the sense that we consider the problems posed in the previous chapter (in this case Chapter 5) with an added pool of players. With the Brouwer fixed point theorem, we provide easy to check conditions for the existence of mean field game equilibrium controls for both the discounted and ergodic control problem and characterize them as the solution of an integro-differential equation. Furthermore, we study the convergence of equilibrium controls in the abelian sense. Finally, we treat the convergence of a finite-player game to this problem to justify our approach (the arguments are exactly the same as Chapter 4).

We incorporate a mean field game dependence into the two-sided discounted and ergodic singular control problems for Lévy processes mentioned in Chapter 5. For both problems we obtain sufficient conditions for the existence of mean field game equilibrium points, and convergence of equilibrium points in the abelian sense. Finally, for a subset of the *admissible controls* we define an N-player problem and prove that a mean-field equilibrium is an approximate Nash equilibrium for the N-player game.

The chapter is organized as follows. In Section 6.2 we define the framework and provide the main results of this chapter. In Section 6.3 we use the adjoint Dynkin game to prove that there is a MFG equilibrium for the ϵ-discounted control problem, for that endeavor we use Brouwer fixed point Theorem. In Section 6.4 we use regenerative theory to prove that the equilibrium points in the discounted case have a convergent subsequence to a MFG equilibrium for the ergodic problem. In Section 6.5 we provide examples. Finally in Section 6.6 we study the convergence of the N-player game to the MFG for both problems.

6.1 Introduction

The mean field game framework when the underlying process is Lévy with related problems as the ones studied in this thesis is scarce. Nevertheless we give some historical references:

- [Fu and Ulrich (2017)] Already mentioned in Chapter 4. The underlying process is an Itô-jump-diffusion.
- [Bensoussan et. al. (2020)] The problem posed here is structurally different from ours, because of this, we omit most of the details in this brief summary. The authors work with a finite-horizon MFG control problem with an integral running and a final cost. The underlying process is the strong solution of a SDE with drift, jump diffusion and regime switching (between finite states). There is a finite number of decision makers $i = 1, \dots n$. Each one has a finite set of controls A_i. The process is controlled by the vector of controls $a = (a_1, \dots, a_n)$. That is, it affects the drift, volatility, jumprate and the switching process. Each decision maker i has a payoff of the form:

$$R_{i,T}(s_0, a, \overline{s}_o) = g_i(s(T), m(T, \cdot), \overline{s}_T) + \int_0^T r_i(s(t), m(t, \cdot), a(t), \overline{s}(t)).$$

  Here, s is the controlled process, s_0 is the departing point, $\overline{s}_0$ is the departing state, m is a probability density depending on the controlled process s (here is where the interaction with infinite players lies). The functions g, r_i are the terminal and integral running cost respectively. The best response is of the form:

$$V_i(0, s_0, \overline{s}_0) = \begin{cases} \sup_{\overline{a}_i \in A_i} \mathbf{E}\left(R_{i,T}(s_0, \overline{a}_i, \overline{s}_0)\right) \\ s(t) \text{ the } a(t) \text{ controlled process starting at } s_0 \text{ with departing state } \overline{s}_0. \\ s(0) = s_0. \\ m(t, \cdot) = \mathbf{P}_{s(t)}. \end{cases} \tag{6.1}$$

  A MFG equilibrium is a vector of controls a such that, for every i the supremum in the first line of (6.1) is reached at a_i. It is interesting to observe that in this case, using naively a HBJ equation would give an infinite-dimensional problem (due to $m(t, \cdot)$). Thus the approach that propose the authors is to use a dual adjoint problem (which is not a Dynkin game, nor an optimal stopping problem). They give sufficient conditions for the existence of a MFG equlibrium and techniques to obtain optimal solutions in the form of a SDE (which its parameters are not explicit in most cases).
- [Benazzoli et al. (2020)] Again, this problem is structurally different from ours as the controls are not singular and the horizon is finite. The authors work with a family of

MFGs with controlled jumps. The controlled process X^γ is of the form:

$$dX_t^\gamma = b(t, X_t^\gamma, \mu_t, \gamma_t)dt + \sigma(t, X_t^\gamma, \mu_t, \gamma_t)dW_t + \beta(t, X_t^\gamma, \mu_t, \gamma_t)d\overline{N}_t,\ t \in [0, T],$$

where μ is a probability measure on the Skorokhod space $D([0, T], \mathbb{R}^d)$ (here lies the mean field interaction) of right-continuous with left limit functions, γ^t represents a control process with values in a fixed action space A, W is a standard multivariate Brownian motion and $\overline{N}$ is a compensated Poisson process with some time-dependent intensity $\lambda(t)$. Moreover, the authors assume that W and $\overline{N}_t$ are independent. There is an integral running cost and a terminal cost. The authors give necessary assumptions to guarantee the existence of a *relaxed mean field game equilibrium*. For that endeavor, they equip the set of *relaxed controls* with the *Wasserstein metric* (see [Carmona and Delarue (2018), Chapter V]) and use a fixed point Theorem. They also provide an example with the explicit solution for the N-player game and show the convergence to the MFG problem.

- [Sohr, T. (2020), Chapter VI, Section 4], the author studies a MFG for an ergodic one-sided impulse control problem with one fixed restarting point y_0 when the underlying process is Lévy with positive mean. The controls are of the form $S = (\tau_n, y_0)$, where $\{\tau_n\}_{n\in\mathbb{N}}$ is a sequence of increasing stopping times. Here the controlled process X^S, informally speaking, is left uncontrolled in the interval $[\tau_n \tau_{n+1})$, that is $X_t^S = X_{\tau_n}^S + X_t - X_{\tau_n}$ for all $t \in [\tau_n \tau_{n+1})$ and in $t = \tau_n$ is pushed to y_0. The controlled process just before the control τ_n is denoted $X_{\tau_n^-}^S$ and is of the form:

$$X_{\tau_n^-}^S = \lim_{t\uparrow\tau_n} X_t^S + \triangle X_{\tau_n}.$$

Moreover, the controls are restricted to the ones such that X^S converge in distribution to a stationary distribution. The control problem is of the form:

$$J_x(R, Q) := \liminf \frac{1}{T}\mathbf{E}_x\left(\sum_{\tau_n\le T}(\varphi(X_{\tau_n^-}^S) - \varphi(y_0))\rho(\mathbf{E}_x(X_\infty^R)) - K\right),\ K > 0.$$

The roadmap is the following:

  - First the authors prove in Chapter IV of this thesis, with an adjoint optimal stopping problem, that the optimal strategies (the best response map when R is fixed) are in the set:

$$\{S = (\{\tau_n^x\}, y_0), \tau_0 := \inf X_t \le x,\ \tau_n = \inf_t X_t^S \le x, \tau_n > \tau_{n+1},\ x > y_0\}$$

– This allows to characterize the strategies (R, Q) with only two parameters (r, q).
– The best response then is a function of one variable.
– The author prove, with the adjoint optimal stopping problem that this map is under the hypothesis of Brouwer fixed point theorem.

Although our framework is structurally different, we follow a similar roadmap.

6.2 Framework and main results

6.2.1 Setting

The probability space with its associated underlying process is the same as Chapter 5 (we assume X is not a subordinator, nor the opposite of a subordinator). Moreover, the set of *admissible controls* $\mathcal{A}$ is also the same. Similar to Section 4.3, in this section, there is a continuous map f and now the function c depends on two variables. Contrary to Chapter 4, we prove that in general there is a MFG equilibrium.

Definition 6.2.1. *We denote $\mathcal{P}^\infty$ the set of random variables X^η_∞ with compact support, such that there exists $\eta = (U, D) \in \mathcal{A}$ so that for every $t \geq 0$:*

$$\lim_{t\to\infty} X_t^{U,D} = X^\eta_\infty, \text{ in distribution.}$$

For simplicity we denote $p^\eta := \mathbf{E}\left(f(X^\eta_\infty)\right)$. Moreover when there is no need to highlight the importance of η we simply denote p^η as p.

Remark 6.2.1. *Let $a < b, x \in \mathbb{R}$, then the probability flux $\mathbf{P}_x(X_t^{a,b} \in dx)$ converges in total variation to a stationary distribution in $\mathcal{P}^\infty$ (see [Andersen et al. (2015), Section V]).*

We define, when $a < b$ the value $p^{a,b} := p^{(U^{a,b}, D^{a,b})}$ and in the degenerate case $a = b$ we define $p^{a,a} := f(a)$.

Proposition 6.2.1. *For every $a \in \mathbb{R}$, the constant random variable $X = a$ belongs to $\mathcal{P}^\infty$.*

Proof. The main idea is to take an smaller and smaller reflection at each time interval. It is clear that it is enough to prove the proposition for $a = 0$ and that we can assume $x = 0$. For $t \in [0, 1]$, let

$$Y_t^n := X_{t+n} - X_n.$$

Observe that $\{Y^n_t\}_{n\geq 0}$ is a sequence of independent Lévy processes starting at zero that satisfy for every n, Y_n is independent of $\sigma(X_u : u \leq n)$. Moreover X_t can be rewritten as:

$$X_t = \sum_{n=0}^{[t]} Y^n_{(t-n)\wedge 1}.$$

We proceed to reflect each process Y^n in $[-1/n, 0]$ in the following way:

$$\begin{aligned}
&U^n_t, D^n_t \text{ the reflections at } [-1/n, 0] \text{ of } Y^n_t,\ t \in [0,1],\ n \geq 0\\
&d^0 = 0, \quad d^n = (Y^{n-1}_n + U^{n-1}_n - D^{n-1}_n)^+,\ n \geq 1,\\
&u^0 = 0, \quad u^n = -(Y^{n-1}_n + U^{n-1}_n - D^{n-1}_n)^-,\ n \geq 1,
\end{aligned}$$

Define the admissible controls U_t, D_t as:

$$U_t := \sum_{n=0}^{[t]} \left(U^n_{(t-n)\wedge 1} + u^n\right),\ D_t := \sum_{n=0}^{[t]} \left(D^n_{(t-n)\wedge 1} + d^n\right).$$

It is clear (U_t, D_t) is an increasing and adapted process and it is satisfied (except in the set that X_t has a jump in a natural number) that $X_t + U_t - D_t \in [-1/n, 0]$ for $t \geq n$. Then we deduce, $\lim_{t\to\infty} X_t + U_t - D_t = 0$ a.s, thus concluding the proof. □

We define the running cost, now depending on $X^\eta_\infty \in \mathcal{P}^\infty$ too, in a way that for every fixed η the map $c(\cdot, \mathbf{E}f(X^\eta_\infty))$ is a running cost as in Definition 5.2.2.

Assumption 6.2.2. *We assume that the function* $c\colon \mathbb{R}^2 \to \mathbb{R}_+$ *is continuous, non-negative and*
(i) *for every* $y \in \mathbb{R}$*, the function* $c(\cdot, y)$ *is strictly convex, has a global minimum in the first variable which is reached at zero and for every* $r > 0$

$$\inf_{(x,y)\in(-r,r)^c\times\mathbb{R}} c_{xx}(x, y) > 0.$$

(ii) *For each fixed* $y \in \mathbb{R}$*, there is a pair of positive constants* N *(independent of* y*) and* M_y *that satisfy*

$$c(x, y) + M_y \geq N|x|,\ \forall x \in \mathbb{R}.$$

(iii) *For every* $(x, y) \in \mathbb{R}^2$

$$\mathbf{E}_x\left(\int_0^\infty c_x(X_s, y)e^{-\epsilon s}ds\right) < \infty,\ \forall x \in \mathbb{R}.$$

Definition 6.2.2. *Given $x \in \mathbb{R}$, $X^{\eta}_{\infty} \in \mathcal{P}^{\infty}$ and a control $(\hat{U}, \hat{D}) \in \mathcal{A}$, we define the ergodic cost function*

$$J(x, (\hat{U}, \hat{D}), X^{\eta}_{\infty}) = \limsup_{T\to\infty} \frac{1}{T}\mathbf{E}_x \left(\int_0^T c(X_s^{\hat{U},\hat{D}}, p^{\eta})ds + q_u\hat{U}_T + q_d\hat{D}_T \right),$$

and the ergodic value function

$$G(x, X^{\eta}_{\infty}) = \inf_{(\hat{U},\hat{D})\in\mathcal{A}} J(x, (\hat{U}, \hat{D}), X^{\eta}_{\infty}).$$

Definition 6.2.3. *Given $x \in \mathbb{R}$, $X^{\eta}_{\infty} \in \mathcal{P}^{\infty}$, a control $(\hat{U}, \hat{D}) \in \mathcal{A}$ and a fixed positive constant ϵ, we define the ϵ-discounted cost function*

$$J_{\epsilon}(x, (\hat{U}, \hat{D}), X^{\eta}_{\infty}) = \mathbf{E}_x \left(\int_0^{\infty} e^{-\epsilon s}(c(X_s^{\hat{U},\hat{D}}, p^{\eta})ds + q_u d\hat{U}_s + q_d d\hat{D}_s \right) + \hat{u}_0 q_u + \hat{d}_0 q_d,$$

and the ϵ-discounted value function

$$G_{\epsilon}(x, X^{\eta}_{\infty}) = \inf_{(\hat{U},\hat{D})\in\mathcal{A}} J_{\epsilon}(x, (\hat{U}, \hat{D}), X^{\eta}_{\infty}).$$

Definition 6.2.4. *We say that a pair of points (a, b) with $a \le b$ (the inequality strict if X has unbounded variation) is :*

(i) *an ϵ-discounted equilibrium if $G_{\epsilon}(x, X^{a,b}_{\infty}) = J_{\epsilon}(x, U^{a,b}, D^{a,b}, X^{a,b}_{\infty})$ for all $x \in \mathbb{R}$,*

(ii) *an ergodic equilibrium if $G(x, X^{a,b}_{\infty}) = J(x, U^{a,b}, D^{a,b}, X^{a,b}_{\infty})$ for all $x \in \mathbb{R}$.*

6.2.2 Main results

The most important results of the chapter are the existence of an equilibrium the discounted problem and the existence of a sequence of ϵ-discounted equilibrium points converging to an ergodic equilibrium point. These results are redacted in the following two Theorems which are proven in the sections 6.3 and 6.4. For $a, b \in \mathbb{R}$, let us recall the definitions of $\tau(a)$ and $\sigma(b)$ given in (5.11):

$$\tau(a) = \inf\{t \ge 0 \colon X_t \le a\}, \quad \sigma(b) = \inf\{t \ge 0 \colon X_t \ge b\}.$$

Theorem 6.2.3. *Under the hypotheses posed in this chapter:*

(i) *for all $X^{\eta}_{\infty} \in \mathcal{P}^{\infty}$, $G_{\epsilon}(\cdot, X^{\eta}_{\infty})$ is in the domain in the infinitesimal generator,*

(ii) *a pair* (a,b) *is an* ϵ*-discounted equilibrium if* $u(x) := G_\epsilon(x, X_\infty^{a,b})$ *satisfies*

$$\begin{aligned}
&\mathcal{L}u(x) - \epsilon u(x) + c(x, p^{a,b}) \leq 0, \text{ for all } x \leq a,\\
&\mathcal{L}u(x) - \epsilon u(x) + c(x, p^{a,b}) \geq 0, \text{ for all } x \geq b,\\
&\mathcal{L}u(x) - \epsilon u(x) + c(x, p^{a,b}) = 0, \text{ for all } x \in (a,b),\\
&u(x) = u(x) + (a-x)q_u, \text{ for all } x \leq a,\\
&u(x) = u(x) + (x-b)q_d, \text{ for all } x \geq b.
\end{aligned} \tag{6.2}$$

(iii) *There is an* ϵ*-discounted equilibrium* $(a_\epsilon^*, b_\epsilon^*)$, *with* $a_\epsilon^* < 0 < b_\epsilon^*$, *that satisfies* ii) *and*

$$\begin{aligned}
V(x, X_\infty^{a_\epsilon^*, b_\epsilon^*}) =& \sup_{a\leq 0} \inf_{b \geq 0} \mathbf{E}_x \left(\int_0^{\tau(a)\wedge\sigma(b)} c_x(X_s, p^{a_\epsilon^*, b_\epsilon^*}) e^{-\epsilon s} ds + q_d e^{-\epsilon \tau(a)} \mathbf{1}_{\tau(a)\leq\sigma(b)} - q_u e^{-\epsilon\sigma(b)} \mathbf{1}_{\sigma(b)<\tau(a)} \right)\\
=& \mathbf{E}_x \left(\int_0^{\tau(a_\epsilon^*)\wedge\sigma(b_\epsilon^*)} c_x(X_s, p^{a_\epsilon^*, b_\epsilon^*}) e^{-\epsilon s} ds + q_d e^{-\epsilon \tau(a_\epsilon^*)} \mathbf{1}_{\tau(a_\epsilon^*)\leq\sigma(b_\epsilon^*)} - q_u e^{-\epsilon\sigma(b_\epsilon^*)} \mathbf{1}_{\sigma(b_\epsilon^*)<\tau(a_\epsilon^*)} \right).
\end{aligned}$$

Furthermore $a_\epsilon^*, b_\epsilon^*$ *satisfy:*

$$a_\epsilon^* = \sup\{x, V(x, X_\infty^{a_\epsilon^*, b_\epsilon^*}) = -q_u\}, \; b_\epsilon^* = \inf\{x, V(x, X_\infty^{a_\epsilon^*, b_\epsilon^*}) = q_d\}.$$

Remark 6.2.2. *The statement* (i) is obtained from Theorem 5.2.1, Proposition 5.4.5 and Proposition 5.4.6. In fact it could have been omitted but we considered it to be informative.

Theorem 6.2.4. *There is an ergodic equilibrium* (a^*, b^*), *with* $a^* \leq 0 \leq b^*$, *that satisfies:*

(i) *There is a sequence* $\{(a_{\epsilon_n}^*, b_{\epsilon_n}^*, \epsilon_n)\}_{n\geq 0}$ *converging to* $(a^*, b^*, 0)$ *when* $n \to \infty$, *such that* $(a_{\epsilon_n}^*, b_{\epsilon_n}^*)$ *is an* ϵ_n*-discounted equilibrium and*

(ii) *for every* $x \in \mathbb{R}$

$$\lim_{n\to\infty} \epsilon_n G_{\epsilon_n}(x, X_\infty^{a_{\epsilon_n}^*, b_{\epsilon_n}^*}) = G(x, X_\infty^{a^* b^*}).$$

6.3 Mean field equilibrium of the ϵ-discounted problem

Due to the fact that for every $y \in \mathbb{R}$, the map $x \to c(x,y)$ is twice countinuously differentiable and with positive bounded below second derivate outside intervals containing zero in the first coordinate, useful properties of the associated Dynkin game are deduced in the first part of this section. Then, in the second part, we use these properties to show that there is an MFG equilibrium with the Brouwer fixed point Theorem.

6.3.1 The adjoint Dynkin game

It is clear that for a fixed $y \in \mathbb{R}$, the results of 5.4 can be applied when the cost function is the map $x \to c(x,y)$ (in this case, as $c(\cdot,y) \in C^2(\mathbb{R})$, there is no need to define the functions c_δ). We borrow the notations from that section with the difference that we add the suffix p in every definition to highlight the dependence of $p = \mathbf{E}\left(f(X^\eta_\infty)\right)$. To be more specific, as these are the only elements of the Dynkin game that will be mentioned in this section, instead of M_x defined in the subsection 5.4.1, we write $M_{x,p}$, instead of V_ϵ defined in the subsection 5.4.2, we write $V_{\epsilon,p}$ and instead of $(a^*_\epsilon, b^*_\epsilon)$ defined in Proposition 5.4.3, we write $(a^*_{\epsilon,p}, b^*_{\epsilon,p})$. Observe however, that Q does not depends on p. Let us recall that in Proposition 5.4.4 we proved that there was an interval $[-L, L]$ such that $(a^*_{\bar\epsilon}, b^*_{\bar\epsilon}) \in [-L, L]$ for every $\bar\epsilon \le \epsilon$. If we were to use this proposition we should highlight the dependence of L with respect to p. However we can refine the proposition so that the bound L does not depends on p. For the reader's convenience, lets recall the definition of γ^ℓ for $\ell > 0$ (see Definition 5.36) :

$$\gamma^\ell = \inf\left\{t \ge 0\colon\ |X_t - \ell| \ge \frac{\ell}{2}\right\}.$$

Proposition 6.3.1. *For ϵ small enough there is a constant L such that the optimal thresholds of the $\bar\epsilon$-Dynkin game $(a^*_{\bar\epsilon,p}, b^*_{\bar\epsilon,p}) \subset [-L, L]$ for every $\bar\epsilon \le \epsilon$ and every $X^\eta_\infty \in \mathcal{P}^\infty$.*

Proof. From Assumptions 6.2.2, for $\ell > 2$ satisfying

$$\ell \left(\inf_{(x,y)\in(-1,1)^c\times\mathbb{R}} c_{xx}(x,y) \right) > N,$$

we have, for all $x \ge \ell/2$, $c_x(x,p) = \int_0^x c_{xx}(u,p)du > N/2$. Then for all $\bar\epsilon > 0$:

$$\mathbf{E}_\ell \left(\int_0^{\gamma^\ell} c_x(X_s, p) e^{-\bar\epsilon s} ds \right) \ge \frac{N}{2} \mathbf{E}_\ell \left(\frac{1 - e^{-\bar\epsilon\gamma^\ell}}{\bar\epsilon} \right). \tag{6.3}$$

On the other hand, as

$$\mathbf{E}_\ell(\gamma^\ell) = \mathbf{E}\left(\inf\left\{ t \ge 0\colon |X_t| \ge \frac{\ell}{2} \right\} \right) \to \infty \text{ as } \ell \to \infty,$$

we find an ℓ s.t.

$$\mathbf{E}_\ell(\gamma^\ell) > \frac{2}{N}(q_u + q_d). \tag{6.4}$$

For a fixed ℓ satisfying (6.4), using dominated convergence and the fact $\mathbf{E}_\ell(\gamma^\ell) < \infty$ when X

is not the null process, we have that

$$\mathbf{E}_\ell\left(\frac{1-e^{-\bar\epsilon\gamma^\ell}}{\bar\epsilon}\right)\nearrow \mathbf{E}_\ell(\gamma^\ell)\quad\text{as } \bar\epsilon\to 0.$$

Thus, using (6.4) we can take ϵ small enough such that for every $\bar\epsilon\le\epsilon$ and every $p\in\mathcal{P}^\infty$ we have

$$\mathbf{E}_\ell\left(\int_0^{\gamma^\ell} c_x(X_s,p)e^{-\bar\epsilon s}ds\right)\ge q_d+q_u. \tag{6.5}$$

We now take $L:=3\ell/2$ and prove that for every $p\in\mathcal{P}^\infty$, $\bar\epsilon\le\epsilon$, $b^*_{\bar\epsilon,p}\le L$. Assume, by contradiction, that $b^*_{\bar\epsilon,p}>L$, what implies $q_d>V_p(\ell)$. Now we have

$$q_d>V_p(\ell)=\mathbf{E}_\ell\left(\int_0^{\gamma_\ell} c_x(X_s,p)e^{-\bar\epsilon s}ds+e^{-\bar\epsilon\gamma_\ell}V_p(X_{\gamma_\ell})\right)\ge\mathbf{E}_\ell\left(\int_0^{\gamma_\ell} c_x(X_s,p)e^{-\bar\epsilon s}ds\right)-q_u\ge q_d.$$

by (6.5), what is a contradiction. The other bound is analogous, thus we obtain an $\bar\ell>0$ such that for every $\bar\epsilon\le\epsilon$ we have $a^*_{\bar\epsilon,}>-3\bar\ell/2$. By taking $L=(3/2)\max(\bar\ell,\ell)$, we conclude the proof. □

Let us recall that from Proposition 5.4.5, we have that V_p is Lipschitz. With that result in mind, we proceed to prove that the ϵ-Nash equilibrium is unique.

Lemma 6.3.2. *If there is a couple (A,B) such that the (τ^A,σ^B) defined as*

$$\tau(A)=\inf\{t\ge 0\colon X_t\le A\},\quad \sigma(B)=\inf\{t\ge 0\colon X_t\ge B\}$$

is a Nash Equilibrium then

$$\liminf_{h\to 0^+}\frac{V_p(x+h)-V_p(x)}{h}\ge\lim_{h\to 0^+}\mathbf{E}\int_0^{\tau(A)_x\wedge\sigma(B)_{x+h}} c_{xx}(x+X_s,p)e^{-\epsilon s}ds. \tag{6.6}$$

Proof. Take $x\in\mathbb{R}$ and $h>0$. We obtain the bound

$$\begin{aligned}V_p(x+h)&\ge\mathbf{E}_{x+h}\left(\int_0^{\tau(A)_{-h}\wedge\sigma(B)} c_x(X_s,p)e^{-\epsilon s}\,ds+Q(\tau(A)_{-h},\sigma(B))\right)\\&=\mathbf{E}\left(\int_0^{\tau(A)_x\wedge\sigma(B)_{x+h}} c_x(x+h+X_s,p)e^{-\epsilon s}ds+Q(\tau(A)_x,\sigma(B)_{x+h})\right).\end{aligned}$$

Similarly

$$V_p(x) \leq \mathbf{E}_x \left(\int_0^{\tau(A)\wedge\sigma(B)_h} c_x(X_s, p) e^{-\epsilon s}\, ds + Q(\tau(A), \sigma(B)_h) \right)$$
$$= \mathbf{E} \left(\int_0^{\tau(A)_x\wedge\sigma(B)_{x+h}} c_x(x + X_s, p) e^{-\epsilon s}\, ds + Q(\tau(A)_x, \sigma(B)_{x+h}) \right).$$

Subtracting, and applying the mean value theorem, we get

$$\frac{V_p(x+h) - V_p(x)}{h} \geq \mathbf{E} \left(\int_0^{\tau(A)_x\wedge\sigma(B)_{x+h}} c_{xx}(x + X_s + \theta h, p) e^{-\epsilon s}\, ds \right),$$

where $0 \leq \theta \leq 1$. We conclude that the inequality (6.6) holds by taking $\liminf_{h\to 0}$ and observing that the limits

$$\lim_{h\to 0} \mathbf{E} \left(\int_0^{\tau(A)_x\wedge\sigma(B)_{x+h}} c_{xx}(x + X_s + \theta h, p) e^{-\epsilon s}\, ds \right),$$
$$\lim_{h\to 0} \mathbf{E} \left(\int_0^{\tau(A)_x\wedge\sigma(B)_{x+h}} c_{xx}(x + X_s, p) e^{-\epsilon s}\, ds \right),$$

exist and are equal due to c_{xx} being continuous and $\sigma(B)_{x+h}$ being monotone in h. □

From the previous lemma we deduce that if there was a couple $(A, B) \neq (a^*_{p,\epsilon}, b^*_{p,\epsilon})$ that was also a Nash equilibrium, we would have different representation for V_p such that the stopping region is different. Thus there would be a point x in one stopping region and in other continuation region. This implies $V_p'(x) > 0$ according to one representation and $V_p'(x) = 0$ according to the other representation (recall V_p' is Lipschitz thus almost everywhere differentiable). We write this result in the next corollary:

Corollary 6.3.3. *The only couple of points (A, B) such that $\tau(A), \sigma(B)$ defined as in the previous Lemma is a Nash Equilibrium is $(a^*_{p,\epsilon}, b^*_{p,\epsilon})$.*

6.3.2 Fixed point

To prove that there is an equlibrium in the discounted case, as usual in the literature, we use a fixed point theorem. To be more specific, we use Brouwer's Fixed Point-Theorem (every continuous function from a compact convex subset of a Euclidean space to itself has a fixed point, see [Park and Sehie (1999), Section 6]). We need some properties before defining the adequate function. In this subsection L is defined as in Proposition 6.3.1. In the following proposition we use implicitly Corollary A.2.4.

Proposition 6.3.4. *For every $r, \epsilon > 0$, $X_\infty^{\mu_1}, X_\infty^{\mu_2} \in \mathcal{P}^\infty$, if*

$$\sup_{x\in[-L,L]} |c'(x, p^{\mu_1}) - c'(x, p^{\mu_2})| \leq r,$$

then

$$||V_{p^{\mu_1}} - V_{p^{\mu_2}}||_\infty \leq r \sup_{x\in[-L,L]} \mathbf{E}_x(\tau_{-L} \wedge \sigma_L),$$

with $V_{p^{\mu_1}}, V_{p^{\mu_2}}$ the (ϵ, p^{μ_1}), (ϵ, p^{μ_2}) value functions respectively.

Proof. Fix $x \in \mathbb{R}$. Observe $V_{p^{\mu_1}}(x) = V_{p^{\mu_2}}(x)$ if $x \notin (-L, L)$. Therefore we can assume $x \in (-L, L)$. Assume $V_{p^{\mu_1}}(x) \leq V_{p^{\mu_2}}(x)$. On one hand, using the fact for $i, j \in \{1, 2\}$, $\tau^*_{a_{p^{\mu_i},\epsilon}} \wedge \sigma^*_{b_{p^{\mu_j},\epsilon}}$ is a stopping time smaller or equal than the first exit of the interval $[-L, L]$:

$$\begin{aligned}
&|M_{x,p^{\mu_1}}(\tau^*_{a_{p^{\mu_i},\epsilon}}, \sigma^*_{b_{p^{\mu_j},\epsilon}}) - M_{x,p^{\mu_2}}(\tau^*_{a_{p^{\mu_i},\epsilon}}, \sigma^*_{b_{p^{\mu_j},\epsilon}})| \\
&\quad = \left|\mathbf{E}_x\left(\int_0^{\tau^*_{a_{p^{\mu_i},\epsilon}} \wedge \sigma^*_{b_{p^{\mu_j},\epsilon}}} (c'(X_s, p^{\mu_1}) - c'(X_s, p^{\mu_2}))e^{-\epsilon s} ds\right)\right| \leq r \sup_{x\in[-L,L]} \mathbf{E}_x(\tau_{-L} \wedge \sigma_L). \quad (6.7)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&V_{p^{\mu_2}}(x) - r \sup_{x\in[-L,L]} \mathbf{E}_x(\tau_{-L} \wedge \sigma_L) \\
&\qquad = M_{x,p^{\mu_2}}(\tau^*_{a_{p^{\mu_2},\epsilon}}, \sigma^*_{b_{p^{\mu_2},\epsilon}}) - r \sup_{x\in[-L,L]} \mathbf{E}_x(\tau_{-L} \wedge \sigma_L) \\
&\qquad \leq M_{x,p^{\mu_2}}(\tau^*_{a_{p^{\mu_2},\epsilon}}, \sigma^*_{b_{p^{\mu_1},\epsilon}}) - r \sup_{x\in[-L,L]} \mathbf{E}_x(\tau_{-L} \wedge \sigma_L) \\
&\qquad \leq M_{x,p^{\mu_1}}(\tau^*_{a_{p^{\mu_2},\epsilon}}, \sigma^*_{b_{p^{\mu_1},\epsilon}}) \leq M_{x,p^{\mu_1}}(\tau^*_{a_{p^{\mu_1},\epsilon}}, \sigma^*_{b_{p^{\mu_1},\epsilon}}) = V_{p^{\mu_1}}(x).
\end{aligned}$$

□

Lemma 6.3.5. *If $\{(a_n, b_n)\}_n$ is a sequence that converges to (a, b), with $a \leq 0 \leq b$, then $p^{a_n,b_n} \to p^{a,b}$ when $n \to \infty$.*

Proof. First we assume $a \neq b$. We can assume that $a_n \neq b_n$ for every n (it is true for n big enough). Then observe, due to (5.47) and the continuity of f:

$$\lim_{n\to\infty} \mathbf{E}\left(f(X_\infty^{a_n,b_n}) - f(X_\infty^{a,b})\right) = \lim_{n\to\infty} \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{E}\left(f(X_s^{a_n,b_n}) - f(X_s^{a,b})\right) ds = 0$$

For the case $a = 0 = b$, simply observe that the measures $X_\infty^{a_n,b_n}$ have support on the interval $[a_n, b_n]$ and $a_n \leq 0 \leq b_n$. $\square$

We need a technical result that will allow us to control the increment of the function V_p

Proposition 6.3.6. *Let us consider the quaternion $b_1 < b_2 < b_3 < b_4$ then*

$$\inf_{x\in[b_2,b_3]} \mathbf{E}_x(\tau_{b_1} \wedge \sigma_{b_4}) > 0.$$

Proof. Assume by contradiction that there is a sequence $\{x_n\} \subset [b_2, b_3]$ converging to x such that

$$\lim_{n\to\infty} \mathbf{E}_{x_n}(\tau_{b_1} \wedge \sigma_{b_4}) = 0.$$

We can assume (taking a subsequence if necessary) that the expected value decreases to zero. Therefore, we deduce $\tau_{b_1-x_n}\wedge\sigma_{b_4-x_n}$ decreases to zero $\mathbf{P}$-almost surely. This implies $\tau_{b_1-x}\wedge\sigma_{b_4-x}$ is equal to zero $\mathbf{P}$ almost surely which is a contradiction because it is the fist exit from an open interval containing zero. $\square$

We have enough properties to work with the map that gives the best response.

Definition 6.3.1. *For a fixed $\epsilon > 0$, define the function $F : [-L, 0] \times [0, L] \to [-L, 0] \times [0, L]$ as: $F(a, b) = (a^*_{p^{a,b},\epsilon}, b^*_{p^{a,b},\epsilon})$. Moreover we denote $F_1(a, b)$ and $F_2(a, b)$ as the projections of F in the first and second coordinate respectively.*

Lemma 6.3.7. *The function F is continuous.*

Proof. We will only prove F_1 is continuous (the proof of the continuity of F_2 is analogue). For $(a, b) \in [-L, 0] \times [0, L]$, take $x \in (a^*_{p^{a,b},\epsilon}, b^*_{p^{a,b},\epsilon})$. Observe $V_{p^{a,b}}(x) < q_d$. Take a sequence $\{(a_n, b_n)\}_{n\geq0} \subset [-L, 0] \times [0, L]$ converging to (a, b). Observe, due to Lemma 6.3.5 and Proposition 6.3.4 we have

$$\limsup_{n\to\infty} V_{p^{a_n,b_n}}(x) < q_d.$$

Therefore $\limsup_{n\to\infty} F_1(a_n, b_n) \geq \liminf_{n\to\infty} F_1(a_n, b_n) \geq b^*_{p^{a,b},\epsilon}$. Let us assume by contradiction that $\limsup_{n\to\infty} F_1(a_n, b_n) = \hat{b} > b^*_{p^{a,b},\epsilon}$. Set $b_1 < b_2$ such that $(b_1, b_2) \subset (b^*_{p_{a,b},\epsilon}, \hat{b})$. Observe, using Lemma 6.3.2 that for N big enough (using the fact $c_{xx}(x, y) > 0$ whenever $x \neq 0$):

$$\liminf_{h\to0^+} \frac{V_{p^{a_n,b_n}}(x+h) - V_{p^{a_n,b_n}}(x)}{h}$$
$$\geq \liminf_{h\to0^+} \mathbf{E} \int_0^{\tau_x^{b^*_{p_{a,b},\epsilon}}\wedge\sigma_{x+h}^{\hat{b}}} \left(\inf_{(h,y)\in(-b^*_{p_{a,b},\epsilon}, b^*_{p_{a,b},\epsilon})^c\times\mathbb{R}} c_{xx}(h, y) \right) e^{-\epsilon s}\, ds.$$

for every $x \in (b_1.b_2)$. Due to the fact

$$\liminf_{h\to 0^+} \sigma_{x+h}^{b_1} \geq \sigma_x^{b_1}, \qquad \inf_{x\in(b_1,b_2)} \mathbf{E}(\tau_x^b \wedge \sigma_x^{\hat{b}}) > 0,$$

we deduce that there is a $\delta > 0$ that for every $x \in (b_1, b_2)$ and n big enough

$$\liminf_{h\to 0^+} \frac{V_{p^{a_n,b_n}}(x+h) - V_{p^{a_n,b_n}}(x)}{h} \\ \geq \mathbf{E}\left(\int_0^{\tau_x^{b^*_{p_{a,b},\epsilon}} \wedge \sigma_x^{\hat{b}}} \left(\inf_{(h,y)\in(-b^*_{p_{a,b},\epsilon}, b^*_{p_{a,b},\epsilon})^c \times \mathbb{R}} c_{xx}(h,y)\right) e^{-\epsilon s}\, ds\right) =: \delta. \quad (6.8)$$

Observe, due to Proposition 6.3.6 the constant δ is greater than zero. Thus, using Lemma 6.3.5 and Proposition 6.3.4, we get:

$$0 = V_{p^{a,b}}(b_2) - V_{p^{a,b}}(b_1) = \lim_{n\to\infty} V_{p^{a_n,b_n}}(b_2) - V_{p^{a_n,b_n}}(b_1) \geq \delta(b_2 - b_1),$$

arriving to a contradiction. Therefore F_1 is continuous. With an analogue argument we can prove F_2 is continuous, concluding the proof of the lemma. □

Proof of Theorem 6.2.3. Due to Proposition 6.3.1 and Lemma 6.3.7, we are in the hypothesis of Brouwer fixed-point Theorem. Thus, using the notation of Proposition 6.3.1 we have a fixed point (a^*, b^*) of the function F. Due to Theorem 5.2.1 we deduce that (a^*, b^*) is an ϵ-discounted equilibrium. □

6.4 Mean field equilibrium for the ergodic problem

To prove Theorem 6.2.4, to show that the optimal strategies converge to an optimal strategy, first we need to show that

$$\lim_{\epsilon\to 0} \epsilon J_\epsilon(x, U^{a,b}, D^{a,b}, p) \to J(x, U^{a,b}, D^{a,b}, p).$$

For that endeavour we need to use theory of regenerative processes (see Section 2.6).

6.4.1 Ergodic results

The main tool that helps us in this section is Theorem 2.6.2 and for that endeavor we need to define an adequate renewal process (see Definition 2.6.1) and an adequate accumulative process (see Definition 2.6.2) We define $\{\tau_n\}_n$ as:

$$\tau_0 = \inf\{t \geq 0, X_t^{0,b} = 0, \sup_{0\leq s\leq t} X_s^{0,b} = b\},$$
$$\tau_{n+1} = \inf\{t \geq \tau_n,\ X_t^{0,b} = 0, \sup_{\tau_n\leq s\leq t} X_s^{0,b} = b\}.$$

Notice that $\{\tau_n\}_n$ is a renewal process (see Lemma 2.6.1). and every τ_n is a stopping time. Moreover, using again Lemma 2.6.1:

$$\mathbf{E}_x(\tau_0) \leq \mathbf{E}_x(\tau_{n+1} - \tau_n) < \infty \text{ for all } n \geq 1,\ x \in \mathbb{R}. \tag{6.9}$$

We proceed to study the abelian limit of the costs.

Proposition 6.4.1. *If X has bounded (unbounded) variation and $a \leq b$ ($a < b$), $x \in \mathbb{R}$, then:*

$$\lim_{\epsilon\to 0} \epsilon J_\epsilon(x, U^{a,b}, D^{a,b}, X_\infty^{a,b}) = J(x, U^{a,b}, D^{a,b}, X_\infty^{a,b})$$

Proof. First of all, notice that the next arguments hold if $a = b$ (when the process has bounded variation). Moreover, it is enough to prove (even if $x \notin [a, b]$):

$$\lim_{\epsilon\to 0} \epsilon \mathbf{E}_x \left(\int_0^{1/\epsilon} c(X_s^{a,b}, p^{a,b}) ds - \int_0^\infty \left(c(X_s^{a,b}, p^{a,b})\right) e^{-\epsilon s} ds \right) = 0, \tag{6.10}$$
$$\lim_{\epsilon\to 0} \epsilon \mathbf{E}_x \left(\int_0^{1/\epsilon} q_u d(U_s^{a,b}) - \int_0^\infty q_u e^{-\epsilon s} d(U_s^{a,b}) \right) = 0, \tag{6.11}$$

To prove (6.10) and (6.11), denote $S_n := \tau_n,\ \forall\ n$. Observe that we can assume $x = a$ (which implies $\tau_0 = 0$ and there is no first jump) due to the strong markov property. Furthermore notice that the processes

$$(Z_1)_t := \int_0^t \left(c(X_s^{a,b}, p^{a,b})\right) ds, \quad (Z_2)_t := \int_0^t q_u d(U_s^{a,b})$$

are cumulative (due to Theorem 2.2.4). We proceed to prove that the three processes are in the hypothesis of Theorem 2.6.2. Firstly, using the continuity of c and (6.9), we deduce Z_1 is in the hypothesis of Theorem 2.6.2.
Secondly notice;

$$\mathbf{E}_a \left(\max_{0\leq t\leq \tau_1} |(Z_2)_t| \right) \leq q_u \mathbf{E}_x U_{\tau_1}^{a,b}.$$

Therefore, to prove that Z_2 is in the hypothesis of Theorem 2.6.2, it is enough to prove

$$\mathbf{E}_a U^{a,b}_{\tau_1} < \infty. \tag{6.12}$$

Observe, due to (5.12) and the fact X has finite mean, it is enough to prove $\mathbf{E}_a(\tau_1) < \infty$. Denoting $\gamma_{b^+}, \ \gamma_{0^-}$ the first time $X^{0,b}$ hits b and 0 respectively, it is clear it is enough to prove

$$\mathbf{E}(\gamma_{b^+}) + \mathbf{E}_b\left(\gamma_{0^-}\right) < \infty. \tag{6.13}$$

Moreover, we only prove (as the other proof follows the same argument) $\mathbf{E}(\gamma_{b^+}) < \infty$ in the appendix, Lemma A.2.3. We deduce (6.13) holds and Z_2 is in the hypothesis of Theorem 2.6.2. Finally to finish the proof of the Proposition, we study the second integral of each equation (6.10) and (6.11). More precisely, observe:

$$\begin{aligned}
&\lim_{\epsilon \to 0} \epsilon \mathbf{E}_a \left(\int_0^\infty c(X^{a,b}_s, p^{a,b}) e^{-\epsilon s} ds \right) = \lim_{\epsilon \to 0} \epsilon \sum_{n=0}^\infty \left(\mathbf{E}_a e^{-\epsilon \tau_1} \right)^n \mathbf{E}_a \left(\int_0^{\tau_1} c(X^{a,b}_s, p^{a,b}) e^{-\epsilon s} ds \right) \\
&= \lim_{\epsilon \to 0} \epsilon \sum_{n=0}^\infty \left(\mathbf{E}_a e^{-\epsilon \tau_1} \right)^n \mathbf{E}_a \left(\int_0^{\tau_1} c(X^{a,b}_s, p^{a,b}) ds \right) = \lim_{\epsilon \to 0} \epsilon \frac{\mathbf{E}_a\left((Z_1)_{\tau_1} \right)}{1 - \mathbf{E}_a(e^{-\epsilon \tau_1})} \\
&= \lim_{\epsilon \to 0} \frac{\mathbf{E}_a\left((Z_1)_{\tau_1} \right)}{\mathbf{E}_a \left(\left(1 - e^{-\epsilon \tau_1} \right) \dfrac{\tau_1}{\epsilon \tau_1} \right)} = \frac{\mathbf{E}_a\left((Z_1)_{\tau_1} \right)}{\mathbf{E}_a \tau_1},
\end{aligned}$$

where in the last equality the dominated convergence Theorem has been used. Therefore, using Theorem 2.6.2, we deduce the equation (6.10) holds. A similar reasoning can be used to deduce that (6.11) hold, concluding the proposition.

$\square$

Lemma 6.4.2. *If X has bounded (unbounded) variation and $a \leq b$ ($a < b$), $\{(a_n, b_n, \epsilon_n)\}_n$ is a sequence that converges to $(a, b, 0)$ when $n \to \infty$ and $a_n \leq b_n$, ($a_n < b_n$) for all n, then:*

$$\lim_{n \to \infty} \epsilon_n J_{\epsilon_n}(x, U^{a_n,b_n}, D^{a_n,b_n}, X^{a_n,b_n}_\infty) = J(x, U^{a,b}, D^{a,b}, X^{a,b}_\infty)$$

for all $x \notin \{a, b\}$.

Proof. First, we assume $x \in (a, b)$. Due to Proposition 6.4.1, it is enough to prove:

$$\lim_{n \to \infty} \epsilon_n \mathbf{E}_x \left(\int_0^\infty \left(c(X^{a,b}_s, p^{a,b}) e^{-\epsilon_n s} \right) ds - \int_0^\infty \left(c(X^{a_n,b_n}_s, p^{a_n,b_n}) \right) e^{-\epsilon_n s} ds \right) = 0 \tag{6.14}$$

and

$$\lim_{n\to\infty} \epsilon_n \mathbf{E}_x \left(\int_0^\infty q_u e^{-\epsilon_n s} d(U_s^{a,b}) - \int_0^\infty q_u e^{-\epsilon_n s} d(U_s^{a_n,b_n}) \right) = 0. \tag{6.15}$$

Equality (6.14) holds due to the fact c is continuous and Lemma 6.3.5.
Equation (6.15) is deduced from Proposition 5.5.2 and integration by parts. For the case $x < a$, observe $x < a_n$ for n big enough. Moreover the initial jump can be ommited in the limit because ϵ_n goes to zero. Finally, by translating the process we observe that it is enough to prove

$$\lim_{n\to\infty} \epsilon_n \mathbf{E} \left(\int_0^\infty \left(c(a + X_s^{0,b-a}, p^{a,b}) e^{-\epsilon_n s}\right) ds - \int_0^\infty \left(c(a_n + X_s^{0,b_n-a_n}, p^{a_n,b_n})\right) e^{-\epsilon_n s} ds \right) = 0$$

and

$$\lim_{n\to\infty} \epsilon_n \mathbf{E} \left(\int_0^\infty q_u e^{-\epsilon_n s} d(U_s^{0,b-a}) - \int_0^\infty q_u e^{-\epsilon_n s} d(U_s^{0,b_n-a_n}) \right) = 0,$$

Following the same line of reasoning as the case $x \in (a, b)$, it can be proven that the three limits hold. Finally the case $x > b$ is obviously analogue to the case $x < a$, thus the proof of the lemma is concluded.

□

6.4.2 Proof of Theorem 6.2.4

We have enough results to prove Theorem 6.2.4.

Proof of Theorem 6.2.4. Using the notation of Proposition 6.3.1, take a sequence $\{(a^*_{\epsilon_n}, b^*_{\epsilon_n})\}_{n\in\mathbb{N}} \subset [-L, L]^2$ such that $(a^*_{\epsilon_n}, b^*_{\epsilon_n})$ is an ϵ_n discounted equilibrium for every n and a couple $(a^*, b^*) \in [-L, L]^2$ satisfying

$$(\epsilon_n, (a^*_{\epsilon_n}, b^*_{\epsilon_n})) \to (0, (a^*, b^*)), \text{ when } n \to \infty.$$

This sequence exists due to Proposition 6.3.1. Now we assume $x \notin \{a^*, b^*\}$. To prove (i) observe it is enough to prove

$$G(x, X_\infty^{a^*,b^*}) - \lim_{n\to\infty} \epsilon_n G_{\epsilon_n}(x, X_\infty^{a^*,b^*}) = 0, \tag{6.16}$$

$$\lim_{n\to\infty} \left(\epsilon_n G_{\epsilon_n}(x, X_\infty^{a^*,b^*}) - \epsilon_n G_{\epsilon_n}(x, X_\infty^{a^*_{\epsilon_n},b^*_{\epsilon_n}}) \right) = 0, \tag{6.17}$$

$$\lim_{n\to\infty} \epsilon_n G_{\epsilon_n}(x, X_\infty^{a^*_{\epsilon_n},b^*_{\epsilon_n}}) - J(x, X_\infty^{a^*,b^*}, U^{a^*,b^*}, D^{a^*,b^*}) = 0. \tag{6.18}$$

The limit (6.16) is deduced from Theorem 5.2.2. To prove that the second limit (6.17) holds, observe that for every $X_\infty^{\mu_1}, X_\infty^{\mu_2} \in \mathcal{P}^\infty, A \le B,\ A, B \in [-L, L], \epsilon > 0$ (the inequality strict if the process has unbounded variation) :

$$\epsilon J_\epsilon(x, U^{A,B}, D^{A.B}, X_\infty^{\mu_1}) - \epsilon J_\epsilon(x, U^{A,B}, D^{A.B}, X_\infty^{\mu_2})$$
$$= \epsilon \mathbf{E}_x \left(\int_0^\infty e^{-\epsilon s} \left(c(X_s^{A,B}, X_\infty^{\mu_1}) - c(X_s^{A,B}, X_\infty^{\mu_2}) \right) ds \right).$$

Therefore:

$$\epsilon_n G_{\epsilon_n}(x, p^{a^*,b^*}) - \epsilon_n G_{\epsilon_n}(x, p^{a^*_{\epsilon_n}, b^*_{\epsilon_n}}) \le 2 \sup_{y \in [-L,L]} |c(y, p^{a^*,b^*}) - c(y, p^{a^*_{\epsilon_n}, b^*_{\epsilon_n}})|. \tag{6.19}$$

Thus from the continuity of c and Lemma 6.3.5, we conclude that the limit (6.17) holds. Finally, the limit (6.18) is deduced from Theorem 5.2.1 and Lemma 6.4.2.
To prove ii), due to i), we use the limits in (6.16) and (6.17).
For the case $x = a^*$, observe G is constant so what we have to study is statement (ii). Take $h > 0$ and observe:

$$\limsup_{n\to\infty} G(a^*, X_\infty^{a^*,b^*}) - \epsilon_n G_{\epsilon_n}(a^*, X_\infty^{a_n^*,b_n^*})$$
$$\le \limsup_{n\to\infty} G(a^*, X_\infty^{a^*,b^*}) - \epsilon_n G_{\epsilon_n}(a^* - h, X_\infty^{a_n^*,b_n^*})$$
$$= G(a^*, X_\infty^{a^*,b^*}) - G(a^* - h, X_\infty^{a^*,b^*}). \tag{6.20}$$

On the other hand

$$\liminf_{n\to\infty} G(a^*, X_\infty^{a^*,b^*}) - \epsilon_n G_{\epsilon_n}(a^*, X_\infty^{a_n^*,b_n^*})$$
$$\ge \limsup_{n\to\infty} G(a^*, X_\infty^{a^*,b^*}) - \epsilon_n G_{\epsilon_n}(a^* - h, p^{a_n^*,b_n^*}) - \epsilon h \left(q_u + q_d\right)$$
$$= G(a^*, X_\infty^{a^*,b^*}) - G(a^* - h, X_\infty^{a^*,b^*}). \tag{6.21}$$

From (6.20) and (6.21) we deduce the theorem for the case $x = a^*$. The case $x = b^*$ is clearly analogue so the proof of the theorem is concluded. □

6.5 Examples

We provide two examples, one for the discounted problem and one for the ergodic problem.

6.5.1 MFG Discounted problem with quadratic cost for a Compound Poisson process with two-sided exponential jumps and Gaussian Noise

This is the extension of the control problem 5.6.3. In this case, $\epsilon > 0$ is fixed, $c(x, y) = h(y)x^2/2$ with h a non-negative continuous function and $f(y)$ a continuous function with infimum greater than zero. Let us recall the Lévy process process $\{X_t\}_{t\geq 0}$ has non-zero mean defined by

$$X_t = x + \sigma W_t + \sum_{i=1}^{N_t^{(1)}} Y_i^{(1)} - \sum_{i=1}^{N_t^{(2)}} Y_i^{(2)}, \tag{6.22}$$

with $\{W_t\}_{t\geq 0}$ a Brownian motion, $\sigma > 0$, and $\{N_t^{(1)}\}_{t\geq 0}$,$\{N_t^{(2)}\}_{t\geq 0}$,$\{Y_i^{(1)}\}_{i\geq 1}$, $\{Y_i^{(2)}\}_{i\geq 1}$ the five processes are independent The control problem is of the form:

$$G_\epsilon(x, \mathbf{E}(f(X_\infty^\eta))) = J_\epsilon(x, (U^{a^*,b^*}, D^{a^*,b^*}), \mathbf{E}(f(X_\infty^\eta))).$$

Similarly to 5.6.3 we have:

$$\begin{aligned} &M_x(a, b, \mathbf{E}(f(X_\infty^\eta))) \\ &\quad = h\left(\mathbf{E}(f(X_\infty^\eta))\right) \mathbf{E}_x\Bigg(\int_0^{\tau(a)\wedge\sigma(b)} e^{-\epsilon s} X_s ds - \frac{q_u}{h\left(\mathbf{E}(f(X_\infty^\eta))\right)} e^{-\epsilon\tau(a)} \mathbf{1}_{\{\tau(a)<\sigma(b)\}} \\ &\qquad\qquad + \frac{q_d}{h\left(\mathbf{E}(f(X_\infty^\eta))\right)} e^{-\epsilon\sigma(b)} \mathbf{1}_{\{\sigma(b)<\tau(a)\}} \Bigg), \end{aligned}$$

and applying Theorem 5.2.1, to solve the discounted control problem, we need to find $a^* < 0 < b^*$ such that

$$M_0(a^*, b^*, \mathbf{E}(f(X_\infty^\eta))) = \sup_{a<0} \inf_{b>0} M_0(a, b, \mathbf{E}(f(X_\infty^\eta))),$$

and then find a fixed point to the adjoint MFG problem. As seen in Example 5.6.3, this can be turned into an analytic problem. With that example in mind the points $a^* \sim -2.017, b^* \sim 2.311$ are a MFG equilibrium for the parameters:

$$q_d = q_u = h\left(\mathbf{E}\left(f(X_\infty^{-2.017,2.311})\right)\right),\ \alpha_1 = 2,\ \alpha_2 = 1,\ \lambda_2 = 1,\ \lambda_1 = 1,\ \epsilon = 1,\ \sigma = \sqrt{2}.$$

6.5.2 Ergodic MFG for Strictly stable process

In this case $c(x, y) = x^2(1 + y)$ and $f(y) = y^2$. This example is the continuation of the example provided in 5.6.4. Here $c(x, y) = x^2 y,\ f(y) = y^2$ and the Lévy process X is strictly α- stable with parameter $\alpha \in (1,2), 0 < c^+ < c^-$. The notations for Π, are the same as 5.6.4. Let us recall

$$\rho = \frac{1}{2} + (\pi\alpha)^{-1} \arctan\left(\left(\frac{c^+ - c^-}{c^+ + c^-}\right)\tan(\alpha\pi/2)\right),$$

from (5.66):

$$\mathbf{E}(D_1^{0,d}) = \frac{c^-\beta(2-\alpha\rho,\alpha\rho) + c^+\beta(2-\alpha(1-\rho),\alpha(1-\rho))}{\beta(\alpha\rho,\alpha(1-\rho))\alpha(\alpha-1)(2-\alpha)}\frac{1}{d^{\alpha-1}} = \mathbf{E}(D_1^{0,1})\frac{1}{d^{\alpha-1}}.$$

and from (5.67) we have

$$a = -\int_0^d x\pi^{0,d}(x)dx = -d\rho,$$

$$J(d,(X_\infty^{0,d}+a)) = (1+\mathbf{E}(X_\infty^{0,d}+a)^2)d^2\frac{\rho(1-\rho)}{\alpha+1} + \frac{1}{d^{\alpha-1}}q\mathbf{E}(D_1^{0,1}).$$

By differentiation:

$$d^* = \left(\frac{(\alpha^2-1)q\mathbf{E}(D_1^{0,1})}{(1+\mathbf{E}(X_\infty^{0,d}-d\rho)^2)2\rho(1-\rho)}\right)^{1/(\alpha+1)}.$$

On the other hand, using (5.65) we deduce

$$1+\mathbf{E}(X_\infty^{0,d}-d\rho)^2 = 1+d^2\frac{\rho(1-\rho)}{\alpha+1}.$$

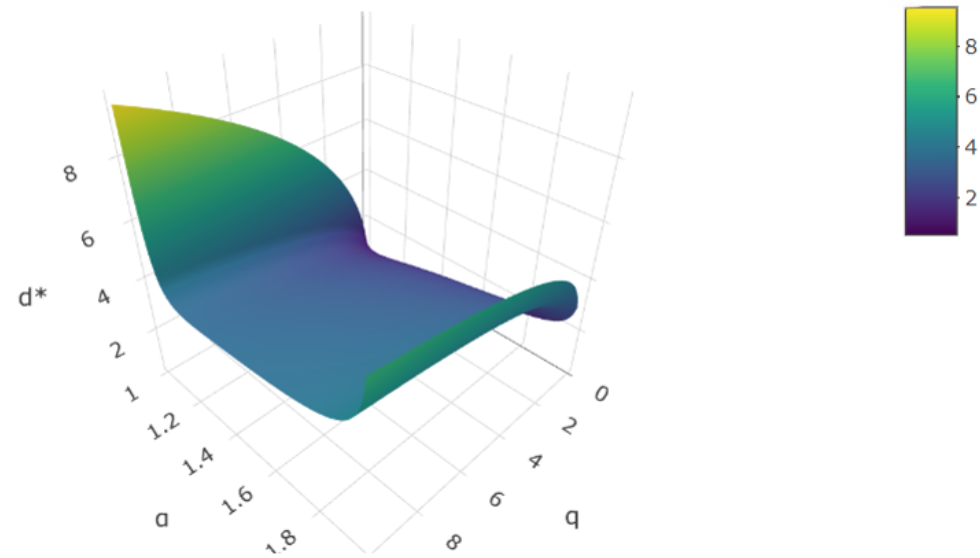
Therefore we conclude that the points (a^*, b^*) that define an ergodic MFG equlibrium are unique and are characterized by the equations:

$$a^* = -b^*\frac{\rho}{1-\rho},$$
$$(b^*/(1-\rho))^{\alpha+1}\left(\frac{\alpha+1+(b^*)^2\rho/(1-\rho)}{\alpha+1}\right) = \frac{(\alpha^2-1)q\mathbf{E}(D_1^{0,1})}{2\rho(1-\rho)}.$$

With the change of variable $u := (b^*)^{\alpha+1}$ we can solve the equation and obtain:

$$a^* = -b^*\frac{\rho}{1-\rho},$$
$$b^* = \left(\frac{-\frac{1}{(1-\alpha)^{\alpha+1}} + \sqrt{\frac{1}{(\alpha-1)^{2\alpha+2}} + 4\frac{\rho}{(1-\rho)^{\alpha+2}}\frac{(\alpha^2-1)q\mathbf{E}(D_1^{0,1})}{2\rho(1-\rho)}}}{2\frac{\rho}{(1-\rho)^{\alpha+2}}}\right)^{1/(\alpha+1)}.$$

For example when we take the values $q = 1,\ c^- = 2,\ c^+ = 1,\ \alpha = 1.5$ the MFG ergodic equilibrium $(a^*, b^*) \sim (-0.52, 0.395)$. We present an illustrative graphic where $c_- = 2, c^+ = 1$, the domain is $(\alpha, q) \in (1,2) \times (0,10)$ and the output is the value $d^* := b^* - a^*$:



6.6 Approximation of Nash equilibria in symmetric N-player games with mean field interaction

In this section, we present an approximation result for Nash equilibria in the N-player game corresponding to the ergodic mean field game considered above, when the number of players N tends to infinity. Informally speaking, this section is very similar to 4, except for the fact that the interaction between players is through its stationary distribution. Moreover we need to work with a more restrictive set of controls. In order to formulate the approximation result, consider:

(i) A filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$ that satisfies the usual conditions, where all the processes are defined.

(ii) Adapted independent Lévy processes $X, \{X^i\}_{i=1,2,\ldots}$.

(iii) Instead of working with the set of admissible controls $\mathcal{A}$, we restrict ourselves to the reflecting controls. In particular we denote $\eta_i^{a,b} = (U^{i,a,b}, D^{i,a,b})$ as the reflection of the process X^i in the barriers $a \leq b$ (the inequality strict if the process has unbounded variation). For simplicity and coherence we denote its $X^{i,a,b}$ controlled process and $X_\infty^{i,a,b}$ its stationary adjoint random variable.

We define a *vector of reflecting barriers* by

$$\Lambda = (\eta_1^{a_1,b_1}, \dots, \eta_N^{a_N,b_N})$$

and similar to Section 4.5,

$$\Lambda^{-i} = (\eta_1^{a_1,b_1}, \dots, \eta_1^{a_{i-1},b_{i-1}}, \eta_1^{a_{i+1},b_{i+1}}, \dots, \eta_N^{a_N,b_N}),$$
$$(\eta^{a,b}, \Lambda^{-i}) = (\eta_1^{a_1,b_1}, \dots, \eta_{i-1}^{a_{i-1},b_{i-1}}, \eta^{a,b}, \eta_{i+1}^{a_{i+1},b_{i+1}}, \dots, \eta_N^{a_N,b_N}),$$

and denote

$$\bar{f}^{-i} = \frac{1}{N-1}\sum_{j\neq i}^{N} f(X_\infty^{j,a_j,b_j}), \quad \bar{f}^{a,b,-i} = \frac{1}{N-1}\sum_{j\neq i}^{N} f(X_\infty^{j,a,b}), \tag{6.23}$$

and, given $\eta = (U^{a,b}, D^{a,b}) \in \mathcal{A}$, for (η, Λ^{-i}), consider

$$J^i_{\infty,N}(x,\eta,\Lambda^{-i}) = \limsup_{T\to\infty} \frac{1}{T}\mathbf{E}_x\left(\int_0^T c\left(X_s^{i,a,b}, \bar{f}^{-i}\right) ds + q_u U_T^{i,a,b} + q_d D_T^{i,a,b}\right), \tag{6.24}$$

$$J_{\epsilon,N}(x,\eta,\Lambda^{-i}) = \mathbf{E}_x\left(\int_0^\infty e^{-\epsilon s}\left(c(X_s^{a,b}, \bar{f}^{-i})ds + q_u dU_s^{a,b} + q_d dD_s^{a,b}\right) + q_u u_0^{a,b} + q_d d_0^{a,b}\right), \tag{6.25}$$

Definition 6.6.1. *For fixed $\epsilon > 0$ and $N \in \mathbb{N}$, a vector of admissible stationary bounded controls $\Lambda = (\eta_1^{a_1,b_1}, \dots, \eta_N^{a_N,b_N})$ is called*

(i) *an r-ergodic Nash equilibrium if for all i and all $x \in \mathbb{R}$,*

$$J^i_{\infty,N}(x, \eta_i^{a_i,b_i}, \Lambda^{-i}) \le J^i_{\infty,N}(x,\mu,\Lambda^{-i}) + r, \quad \textit{for all } \mu \textit{ reflecting control.}$$

(ii) *an r,ϵ-discounted Nash equilibrium if for all i and all $x \in \mathbb{R}$,*

$$J^i_{\epsilon,N}(x, \eta_i^{a_i,b_i}, \Lambda^{-i}) \le J^i_{\epsilon,N}(x,\mu,\Lambda^{-i}) + r, \quad \textit{for all } \mu \textit{ reflecting control.}$$

We omit the proof of the next theorem as it is almost equal to the proof of the first statement of Theorem 4.5.1 (clearly in the discounted case the integral is in the half-positive line and there is an exponential factor but the proof is the same).

Theorem 6.6.1. *Consider a cost function $c(x,y)$ that satisfies Assumption 6.2.2, for every fixed x the function $c(x,y)$ is convex and the set of admissible controls is the set of reflecting controls for each process X^i, $i = 1,\dots,N$, instead of $\mathcal{A}$, Then, if (a,b) is an equilibrium point* *for the mean field game driven by X, given $r > 0$, the vector of controls*

$$\Lambda^{a,b} = ((U^{1,a,b}, D^{1,a,b}), \ldots, (U^{N,a,b}, U^{N,a,b})), \tag{6.26}$$

is

(i) *a r-ergodic Nash equilibrium for N large enough.*

(ii) *a r, ϵ-discounted Nash equilibrium for N large enough.*

Let us observe that, unlike Chapter 4, the case where the cost function is $c(x, y) = |x - y|$ is not treated in this theorem as it is not a running cost function under Definition 6.2.2.

A.1 Reflection of Itô diffusions on intervals

The objective of this Appedix is to give a simplified proof of [Lions and Sznitman (1984), Theorem 3.1]. That is to show that for an Itô-diffusion the reflecting controls are well defined in the sense that they exist and are unique.

Remark A.1.1. *In the article [Lions and Sznitman (1984)], the authors assumed the coefficients μ, σ to be globally Lipschitz. However in our framework where the reflection is in an interval, this hypothesis can be easily relaxed to locally Lipschitz.*

The notations and hypotheses are the same as chapters 3 and 4. Let us recall from 2.5 the following definition

Definition A.1.1. *Let $a < b$ be a pair of real numbers. The* double Skorkhod map $\Gamma_{a,b}$ *is the mapping from $\mathbb{D}[0,\infty)$ into itself such that for $\rho \in \mathbb{D}[0,\infty)$, $\Gamma_{a,b}(\rho)$ takes values in $[a,b]$ and has the decomposition.*

$$\Gamma_{a,b}(\rho) = \rho + U^{a,b} - D^{a,b},$$

where $U^{a,b}, D^{a,b} \in \mathbb{D}[0,\infty)$ are non decreasing and satisfy

$$\int_0^\infty (\Gamma_{a,b}(\rho)(t) - a) dU_t^{a,b} = 0, \quad \int_0^\infty (b - \Gamma_{a,b}(\rho)(t)) dD_t^{a,b} = 0.$$

Moreover, in $t = 0$ the functions $U^{a,b}, D^{a,b}$ project $\rho(0)$ to the closest point in $[a,b]$.

As said in 2.5 for every càdlàg function and pair $a < b$ the *double Skorkhod map* is well defined in the sense that it exists and is unique. We will use this result in the main Theorem of this section. For the rest of the section $a < b$ are fixed and μ and $\sigma > 0$ are locally Lipschitz functions. We also denote the deterministic reflections of a process $Z = \{Z_t\}_{t\geq 0}$ on $[a,b]$ as $U_Z^{a,b} = \{U_{Z,t}^{a,b}\}_{t\geq 0}$, $D_Z^{a,b} = \{D_{Z,t}^{a,b}\}_{t\geq 0}$.

Definition A.1.2. *We denote the space H to be the Frechet space of continuous adapted processes X satisfying*

$$\mathbf{E} \sup_{0\leq s\leq t} X_t^4 < \infty, \quad \textit{for all } t \geq 0,$$

eqquiped with the seminorm

$$||X||_t = \left(\mathbf{E} \sup_{0\leq s\leq t} X_t^4 \right)^{1/4}.$$

Moreover we define $F : H \to H$ in the following way:

- *Take the process $Z = \{Z_t\}_{t\geq 0}$ as follows:*

$$dZ_t = \mu(X_t)dt + \sigma(X_t)dW_t, \; Z_0 = x_0. \; \textit{ with } x_0 \; \textit{ the starting point of } X.$$

- *Define $Y = \{Y_t\}_{t\geq 0}$ as the controlled process:*

$$Y_t = Z_t + U^{a,b}_{Z,t} - D^{a,b}_{Z,t}$$

It is clear then, that we need to find a fixed point of F and prove its uniqueness.

Lemma A.1.1. *Assume μ and σ are globally Lipschitz, then there are four positive constants $K_1,\ K_2,\ K_3,\ K_4$ satisfying*

$$||F(X) - F(X')||_T^4 \leq e^{K_1 T + K_2 T^2}(K_3 + K_4 T)\int_0^T ||X - X'||_s^4 ds, \quad \textit{for every } T \geq 0,\ X, X' \in H.$$

Proof. By Itô formula we have for every $t \geq 0$:

$$\begin{aligned}(Y_t - Y_t')^2 &= 2\int_0^t (Y_s - Y_s')\left(\sigma(X_s) - \sigma(X_s')\right) dW_s + 2\int_0^t (Y_s - Y_s')\left(\mu(X_s) - \mu(X_s')\right) ds \\ &\quad + \int_0^t \left(\sigma(X_s) - \sigma(X_s')\right)^2 ds + 2\int_0^t (Y_s - Y_s')\left(dU^{a,b}_{Z,t} - dD^{a,b}_{Z,t} - dU^{a,b}_{Z',t} + dD^{a,b}_{Z',t}\right).\end{aligned}$$

The last term is non-positive because the reflection only increase when the process is in the barrier. Thus we have the inequality:

$$\begin{aligned}(Y_t - Y_t')^2 &\leq 2\int_0^t (Y_s - Y_s')\left(\sigma(X_s) - \sigma(X_s')\right) dW_s + 2\int_0^t \left|(Y_s - Y_s')\left(\mu(X_s) - \mu(X_s')\right)\right| ds \\ &\quad + 2\int_0^t \left(\sigma(X_s) - \sigma(X_s')\right)^2 ds =: I_t. \end{aligned} \tag{27}$$

Observe that the integral with respect to the brownian motion is a martingale due to the fact that σ is Lipschitz, Y, Y' are bounded and

$$\int_0^t (X_s - X_s')^2 ds < \infty.$$

Therefore we have that the process

$$\left(\int_0^t (Y_s - Y_s')\left(\sigma(X_s) - \sigma(X_s')\right) dW_s\right)^2 - \int_0^t (Y_s - Y_s')^2\left(\sigma(X_s) - \sigma(X_s')\right)^2 ds \tag{28}$$

is a local martingale. Using [Protter (2005), Chapter II, Setion 6, Corollary 3], we can take means in (33) to obtain

$$\mathbf{E}\left(\int_0^t (Y_s - Y_s')\left(\sigma(X_s) - \sigma(X_s')\right) dW_s\right)^2 = \mathbf{E}\int_0^t (Y_s - Y_s')^2 \left(\sigma(X_s) - \sigma(X_s')\right)^2 ds, \tag{29}$$

for all $t \geq 0$. Therefore I_t is a submartingale. Moreover, using Cauchy inequality, (27) and (29) we deduce:

$$\begin{aligned}\mathbf{E}I_t^2 \leq 12\mathbf{E}\int_0^t (Y_s - Y_s')^2 \left(\sigma(X_s) - \sigma(X_s')\right)^2 ds + 12t\mathbf{E}\int_0^t (Y_s - Y_s')^2 \left(\mu(X_s) - \mu(X_s')\right)^2 ds \\ + 3t\mathbf{E}\int_0^t \left(\sigma(X_s) - \sigma(X_s')\right)^4 ds. \end{aligned} \tag{30}$$

Now, we use Doob's inequality, the fact μ, σ are Lipschitz and (30) to deduce there are three of positive constants $C_1,\ C_2,\ C_3$, such that:

$$\begin{aligned}\mathbf{E}\left(\sup_{0\leq s\leq t}(Y_t - Y_t')^4\right) &\leq \mathbf{E}\sup_{0\leq s\leq t} I_s^2 \\ &\leq 4\mathbf{E}I_t^2 \leq (C_1 + C_2 t)\int_0^t \mathbf{E}(Y_s - Y_s')^2(X_s - X_s')^2 ds + C_3 t\int_0^t \mathbf{E}(X_s - X_s')^4 ds.\end{aligned}$$

By using the inequality $xy \leq \frac{x^2+y^2}{2}$:

$$\begin{aligned}\mathbf{E}\left(\sup_{0\leq s\leq t}(Y_t - Y_t')^4\right) &\leq \mathbf{E}\sup_{0\leq s\leq t} I_s^2 \leq 4\mathbf{E}I_t^2 \\ &\leq \frac{C_1 + C_2 t}{2}\int_0^t \mathbf{E}(Y_s - Y_s')^4 ds + \frac{C_1 + C_2 t + 2C_3 t}{2}\int_0^t \mathbf{E}(X_s - X_s')^4 ds.\end{aligned}$$

Therefore, by renaming the constants, for every $t \leq T$:

$$\mathbf{E}\left(\sup_{0\leq s\leq t}(Y_t - Y_t')^4\right) \leq (K_1 + K_2 T)\int_0^t \mathbf{E}(Y_s - Y_s')^4 ds + (K_3 + K_4 T)\int_0^t \mathbf{E}(X_s - X_s')^4 ds.$$

Finally, by using Grownall lemma applied to the function $t \to \mathbf{E}\left(\sup_{0\leq s\leq t}(Y_t - Y_t')^4\right)$:

$$\mathbf{E}\left(\sup_{0\leq s\leq t}(Y_t - Y_t')^4\right) \leq e^{K_1 T + K_2 T^2}(K_3 + K_4 T)\int_0^t \mathbf{E}(X_s - X_s')^4 ds,$$

we conclude the Lemma. □

Theorem A.1.2. *Adaptation of [Lions and Sznitman (1984), Theorem 3.1] Let $X = \{X_t\}_{t\geq 0}$ be an Itô-diffusion with associated Lipschitz functions μ and σ. Let $a < b$ and $x \in \mathbb{R}$, then there* *exists an unique triple* $(X^{a,b} = \{X_t^{a,b}\}_{t\geq 0}, U^{a,b} = \{U_t^{a,b}\}_{t\geq 0}, D^{a,b} = \{D_t^{a,b}\}_{t\geq 0})$ *such that* $X^{a,b}$ *is the unique strong solution of the SDE*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dU_t^{a,b} - dD_t^{a,b}, \;\; X_0 = x, \;\; U_0^{a,b} = (a-x)^+, \;\; D_0^{a,b} = (x-b)^+, \quad (31)$$

and $U^{a,b}, D^{a,b}$ *are increasing continuous processes satisfying*

$$\int_0^\infty (X_t^{a,b} - a)dU_t^{a,b} = 0, \qquad \int_0^\infty (b - X_t^{a,b})dD_t^{a,b} = 0.$$

Proof. Due to the definitions of $U_0^{a,b}$ and $D_0^{a,b}$ we can assume the starting point $x \in [a,b]$. It is clear that it is enough to prove that F has an unique fixed point. For a fixed $X_0 \in H$, let $X^{(n)} := F \circ \cdots \circ F(X) = F^{(n)}(X), \; n \geq 1$. First we prove by induction that the next inequality holds:

$$\mathbf{E} \sup_{0\leq s\leq T} (X_s^{n+1} - X_s^n)^4 \leq \frac{e^{n(K_1 T + K_2 T^2)}(K_3 + K_4 T)^n (b-a)^4}{n!}, \quad \text{for all } T \geq 0, n \geq 1. \qquad (32)$$

The case base $n = 1$ holds due to Proposition A.1.1. Now, we assume the claim holds for $n \geq 1$ and again we use Proposition A.1.1 to get :

$$\begin{aligned}\mathbf{E} \sup_{0\leq s\leq T} (X_s^{n+2} - X_s^{n+1})^4 &\leq e^{K_1 T + K_2 T^2}(K_3 + K_4 T) \int_0^T \mathbf{E}\left(\sup_{0\leq u\leq s} (X_u^{n+1} - X_u^n)^4\right) ds \\ &\leq e^{K_1 T + K_2 T^2}(K_3 + K_4 T) \int_0^T \frac{s^n e^{n(K_1 T + K_2 T^2)}(K_3 + K_4 T)^n (b-a)^4}{n!} ds \\ &= \frac{e^{(n+1)(K_1 T + K_2 T^2)}(K_3 + K_4 T)^{n+1}(b-a)^4}{(n+1)!}.\end{aligned}$$

By Chebyshev inequality we have:

$$\mathbf{P}\left(\sup_{0\leq s\leq T} |X_s^n - X_s^{n+1}|^2 > \frac{1}{n^2}\right) \leq \frac{n^2 e^{n(K_1 T + K_2 T^2)}(K_3 + K_4 T)^n (b-a)^4}{n!}.$$

Now, we use Borel Cantelli Lemma (see [Borodin (2013), page 133], for a similar argument) and deduce there is a continuous bounded adapted process $X \in H$ such that

$$\mathbf{P}\left(\lim_{n\to\infty} \sup_{0\leq s\leq T} |X_s^n - X_s| = 0\right) = 1, \qquad \lim_{n\to\infty} ||X^n - X||_T = 0, \;\; \text{for all } T \geq 0.$$

Due Lemma A.1.1, we can use the $||\;||_T$ continuity of the map F and conclude X is a fixed point, concluding the proof of the existence. For the uniqueness, take X, Y two fixed points of

F. As in (32) we can prove:

$$\mathbf{E}\sup_{0\le s\le T}(F^n(X)-F^n(Y))^4 \le \frac{e^{n(K_1T+K_2T^2)}(K_3+K_4T)^n(b-a)^4}{n!}, \quad \text{for all } T\ge 0,\ n\ge 1,$$

which clearly implies $X=Y$ in $[0,T]$ almost surely as n is arbitrary. □

Remark A.1.2. *It is clear that we only proved the existence and uniqueness in $[0,T]$. However this implies both results in all the real half-line as one can observe that if X_1 and X_2 are the solutions of the SDE in Theorem A.1.2 in $[0,T_1]$ and $[0,T_2]$ respectively with $T_1<T_2$ then $X_2=X_1$ in $[0,T_1]$ (see [Protter (2005), Chapter V, Section 3, Theorem 7], for a similar argument).*

Corollary A.1.3. *The same result as the previous Theorem is valid if μ and σ are locally Lipschitz.*

Proof. For both the existence and uniqueness, take $\overline{\mu}$, $\overline{\sigma}$ a couple of globally Lipschitz functions whose restriction in $[a,b]$ are equal to μ and σ respectively and use Theorem A.1.2. □

A.2 Regenerative properties of Lévy processes

In this section the objective is to prove the technical ergodic and probabilistic results for the controlled Lévy processes we assumed in the previous chapters. We use the same notations and hypotheses as Chapters 5 and 6.

Lemma A.2.1. *If $Z=\{Z_t\}_{t\ge 0}$ is a bounded semimartingale and $X=\{X_t\}_t$ is a Lévy process with zero mean then:*

$$I_t := \int_0^t Z_{s^-}dX_s,\ t\ge 0, \qquad \textit{is a martingale.}$$

Proof. From [Protter (2005), Chapter IV, Section 2, Theorem 11], we have that the result holds if X is a square-integrable martingale. Therefore it is enough to prove the result when X is a finite mean, Compound Poisson process with drift μ. That is, there is a i.i.d sequence $\{Y_i\}_{i\in\mathbb{N}}$ of finite mean, random variables and a poisson process $N=\{N_t\}_{t\ge 0}$ with intensity $\lambda>0$ such that:

$$X_s = x+\sum_{i=1}^{N_t} +\mu t$$

We know from [Protter (2005), Chapter IV, Section 2, Theorem 11], $Z_{s^-}dX_s$ is a local martingale, that is, there exist an increasing sequence $\{\tau_n\}_{n\ge 0}$ of stopping times converging to ∞

almost surely such that $Z_{s^-} dX_s^{\tau_n}$ is a martingale. Take $t \geq u$, we have:

$$\lim_{n\to\infty} \mathbf{E}\left(\int_0^{t\wedge\tau_n} Z_{s^-} dX_s \middle| \mathcal{F}_u\right) = \lim_{n\to\infty} \int_0^{u\wedge\tau_n} Z_{s^-} dX_s. \tag{33}$$

On the other had, we have for all $n \in \mathbb{N}, t > 0$:

$$\left|\int_0^t Z_{s^-} dX_s\right| \leq \sup_{s\geq 0} |Z_{s^-}| \left(\sum_{i=1}^{N_t} |Y_i| + |\mu| t\right).$$

Thus we can use dominated convergence theorem in both sides of equality (33) to conclude the lemma.

□

Proposition A.2.2. *Assume $X = \{X_t\}_{t\geq 0}$ is a finite mean Lévy process which is not trivial, nor a subordinator, nor the opposite of a subordinator. For a fixed $b > 0$, by writing X as the following sum of independent Lévy processes:*

$$X_t = \mu t + \overline{N}_t + \int_{|x|\geq\delta} \int_0^t xN(ds \times dx) + \sigma W_t, \qquad \overline{N} \text{ a martingale, } \delta > 0$$

we have:

(i) *If $\mu \leq 0$ then there is a $t > 0$ such that*

$$\mathbf{P}\left(\sup_{0\leq s\leq t} X_s \geq b\right) > 0.$$

(ii) *In other case the same result is valid for every $t \geq 3b/\mu$.*

Proof of (i). If $\sigma = 0$, we can assume that $\pi[\delta, \infty) > 0$. Using the fact that Lévy processes tend to zero when $t \to 0^+$ almost surely, we deduce there is a t small enough such that:

$$\mathbf{P}\Bigg(\left(\mu t + \overline{N}_t + \int_{x<-\delta} \int_0^t xN(ds \times dx) > -b\right) \\ \bigcap\left(\left(\sup_{0\leq s\leq t} \sigma W_s \geq 2b\right) \bigcup \left(\int_{x>\delta} \int_0^t xN(ds \times dx) \geq 2b\right)\right)\Bigg) > 0,$$

because the three processes are independent and the last one is a Compound Poisson process.

□

Proof of (ii). For the second proof, simply observe that for every $t > 0$

$$\mathbf{P}\left(\inf_{0\le s\le 3b/\mu} X_s - \mu s > -b\right) > 0$$

□

Lemma A.2.3. *Let $X = \{X_t\}_{t\ge 0}$ be a Lévy process with finite mean such that X is not trivial, is not a subordinator nor the opposite of a subordinator. Let $\{\tau_n\}_n$ be defined as:*

$$\begin{aligned}\tau_0 &= \inf\{t \ge 0, X_t^{0,b} = 0, \sup_{0\le s\le t} X_s^{0,b} = b\},\\ \tau_{n+1} &= \inf\{t \ge \tau_n,\ X_t^{0,b} = 0, \sup_{\tau_n\le s\le t} X_s^{0,b} = b\}.\end{aligned}$$

Then $\{\tau_n\}_{n\in\mathbb{N}}$ is a renewal process and $\mathbf{E}(\tau_n) < \infty$ for every $n \in \mathbb{N}$.

Proof. Due to the strong markov property, the only non trivial statement is that the stopping times have finite mean. It is clear $\mathbf{E}\tau_1 = \mathbf{E}(\tau_n - \tau_{n-1})$ for all $n \ge 1$ and $\mathbf{E}\tau_1 \le \mathbf{E}\tau_0$. Thus it is enough to prove $\mathbf{E}\tau_0 < \infty$. Denoting $\gamma_{b^+},\ \gamma_{0^-}$ the first time $X^{0,b}$ hits b and 0 respectively, it is clear it is enough to prove

$$\mathbf{E}(\gamma_{b^+}) + \mathbf{E}_b\left(\gamma_{0^-}\right) < \infty.$$

We only prove $\mathbf{E}(\gamma_{b^+}) < \infty$ as the other claim is analogue. Take $t > 0$ as in Proposition A.2.2, then we have:

$$\mathbf{P}\left(\sup_{0\le s\le t} X_s^{0,b} < b\right) \le \mathbf{P}\left(\sup_{0\le s\le t} X_s < b\right) < 1. \tag{34}$$

On the other hand observe for all $i \ge 1$:

$$\left\{\omega : \sup_{t(i-1)\le s\le ti} X_s^{0,b} < b\right\} \subset \left\{\omega : \sup_{t(i-1)\le s\le ti} X_s - X_{t(i-1)} < b,\right\},$$

because if $X_s - X_{t(i-1)} \ge b$ for some $s \in [t(i-1), ti]$ then there has been an increment equal or bigger than b in that interval which implies $\sup_{t(i-1)\le s\le t_i} X_s^{0,b} = b$. We deduce:

$$\begin{aligned}\mathbf{P}\left(\sup_{0\le s\le tn} X_s^{0,b} < b\right) &= \mathbf{P}\left(\bigcap_{i=1}^{n} \sup_{t(i-1)\le s\le ti} X_s^{0,b} < b\right)\\ &\le \prod_{i=1}^{n} \mathbf{P}\left(\sup_{t(i-1)\le s\le ti} X_s - X_{t(i-1)} = b\right) \le \mathbf{P}\left(\sup_{0\le s\le t} X_s < b\right)^n.\end{aligned} \tag{35}$$

Therefore, using inequalities (34) and (35) we conclude:

$$\mathbf{E}(\gamma_{b^+}) = \int_0^\infty \mathbf{P}(\gamma_{b^+} > x)dx \le \sum_{n=1}^{\infty} t\mathbf{P}(\gamma_{b^+} > t(n-1)) \le t\sum_{n=1}^{\infty} \mathbf{P}\left(\sup_{0\le s\le t} X_s < b\right)^{n-1} < \infty.$$

□

Corollary A.2.4. *Consider a pair of constants $a < b$, the following stopping time has finite mean:*

$$\eta_{a,b} := \inf\{t \ge 0, X_t \notin (a,b)\}.$$

A.3 The problem of the first exit time of an interval for some Lévy processes

For a Lévy process $X = \{X_t\}_{t\ge 0}$, and a given couple $a < 0 < b$ and η defined as the first exit of the interval (a,b), the *two-barrier problem* consists in finding, the probabilities:

$$\mathbf{P}(X_\eta < a), \qquad \mathbf{P}(X_\eta = a), \qquad \mathbf{P}(X_\eta = b), \qquad \mathbf{P}(X_\eta > b),$$

and for $\epsilon > 0$ the *generalized discounted two-barrier problem*

$$\mathbf{E}e^{-\epsilon\eta}\mathbf{1}_{X_\eta<a}, \qquad \mathbf{E}e^{-\epsilon a}\mathbf{1}_{X_\eta=a}, \qquad \mathbf{E}e^{-\epsilon b}\mathbf{1}_{X_\eta=b}, \qquad \mathbf{E}e^{-\epsilon\eta}\mathbf{1}_{X_\eta>b}.$$

These problems are open for general Lévy processes. The (discounted) two-barrier problem for the processes given in the examples provided in chapters 5 and 6 were solved in [Kyprianou(2006)] and [Cai et al. (2009)]. Nevertheless, as the references for Compound Poisson process with two-sided exponential jumps with and without gaussian noise are more general than our cases, we give a simplified solution of the problem for our particular case. We can assume $X_0 = 0$ because:

$$\mathbf{P}_x(X_\gamma < a) = \mathbf{P}(X_{\gamma^x} < a-x),\ \mathbf{P}_x(X_\gamma \le a) = \mathbf{P}(X_{\gamma^x} \le a-x),\ \mathbf{P}_x(X_\gamma \ge b) = \mathbf{P}(X_{\gamma^x} \ge b-x),$$

with γ^x the first exit of $(a-x, b-x)$. A powerful tool that we use in this section is the function

$$\phi(z) = \mathbf{E}(e^{zX_1}),\ \ z = i\theta \in i\mathbb{R}.$$

A.3.1 Two-barrier problem for Poisson Compound Process with two-sided exponential jumps and negative mean

We consider a compound Poisson process $X = \{X_t\}_{t\geq 0}$ with double-sided exponential jumps, given by

$$X_t = \sum_{i=1}^{N_t^{(1)}} Y_i^{(1)} - \sum_{i=1}^{N_t^{(2)}} Y_i^{(2)}, \tag{36}$$

where $\{N_t^{(1)}\}_{t\geq 0}$ and $\{N_t^{(2)}\}_{t\geq 0}$ are two Poisson processes with respective positive intensities λ_1, λ_2; $\{Y_i^{(1)}\}_{i\geq 1}$ and $\{Y_i^{(2)}\}_{i\geq 1}$ are two sequences of independent exponentially distributed random variables with respective positive parameters α_1, α_2. The four processes are independent. Consequently

$$\phi(z) = \lambda_1 \frac{z}{\alpha_1 - z} - \lambda_2 \frac{z}{\alpha_2 + z}.$$

In this case $\mathbf{E}X_1 = \lambda_1/\alpha_1 - \lambda_2/\alpha_2 < 0$.

Lemma A.3.1. *Consider the Lundberg constant ρ, i.e the positive root of $\phi(z) = 0$, given by*

$$\rho = \frac{\lambda_2\alpha_1 - \lambda_1\alpha_2}{\lambda_1 + \lambda_2}.$$

For $a < 0 < b$

$$\mathbf{P}(X_\eta < a) = \frac{\left(\dfrac{\alpha_1 + \alpha_2}{\lambda_1 + \lambda_2}\right) - e^{\rho b}/\lambda_1}{e^{\rho a}/\lambda_2 - e^{\rho b}/\lambda_1}, \qquad \mathbf{P}(X_\eta > b) = \frac{e^{\rho a}/\lambda_2 - \left(\dfrac{\alpha_1 + \alpha_2}{\lambda_1 + \lambda_2}\right)}{e^{\rho a}/\lambda_2 - e^{\rho b}/\lambda_1}.$$

Proof. Observe that $0 < \rho < \alpha_1$. Therefore, the integrals:

$$\mathbf{E}(e^{\rho X_\eta}\mathbf{1}_{X_\eta < a}), \qquad \mathbf{E}(e^{\rho X_\eta}\mathbf{1}_{X_\eta > b}),$$

are well defined and $\{e^{\rho X_t}\}_{t\geq 0}$ is a martingale. Using the loss of memory property of the jumps we get the system of equations

$$\begin{aligned}
1 &= \mathbf{P}(X_\rho < a)\int_{-\infty}^{0} e^{\alpha_2 u} e^{\rho(u+a)}\alpha_2 du + \mathbf{P}(X_\rho > b)\int_{0}^{\infty} e^{-\alpha_1 u} e^{\rho(u+b)}\alpha_1 du,\\
1 &= \mathbf{P}(X_\rho < a) + \mathbf{P}(X_\rho > b).
\end{aligned}$$

Which is equivalent to the system:

$$\begin{aligned}
1 &= e^{\rho a}\mathbf{P}(X_\rho < a)\frac{(\lambda_1+\lambda_2)/\lambda_2}{\alpha_1+\alpha_2} + e^{\rho b}\mathbf{P}(X_\rho > b)\frac{(\lambda_1+\lambda_2)/\lambda_1}{\alpha_1+\alpha_2},\\
1 &= \mathbf{P}(X_\rho < a) + \mathbf{P}(X_\rho > b).
\end{aligned}$$

This system has rank 2 because

$$e^{\rho a}\frac{(\lambda_1+\lambda_2)/\lambda_2}{\alpha_1+\alpha_2} = e^{\rho a}\frac{\alpha_2}{\rho+\alpha_2} < 1.$$

Moreover, taking into account the fact

$$\frac{\alpha_1}{\alpha_1-\rho} = \frac{(\lambda_1+\lambda_2)/\lambda_1}{(\alpha_1+\alpha_2)},$$

we deduce the solution is the one given in (37), thus concluding the proof of the lemma. □

A.3.2 Discounted two-barrier problem for a Compound Poisson process with two sided exponential jumps, Gaussian Noise and non-zero mean

This subsection is an adaptation of [Cai et al. (2009)] for our particular case. Still, we prove all the results to capture the essence of the paper.

Let ϵ be a positive fixed constant. The Lévy process process $\{X_t\}_{t\geq 0}$ has non-zero mean defined by

$$X_t = \sigma W_t + \sum_{i=1}^{N_t^{(1)}} Y_i^{(1)} - \sum_{i=1}^{N_t^{(2)}} Y_i^{(2)},$$

with $\{W_t\}_{t\geq 0}$ a Brownian motion, $\sigma > 0$, and $\{N_t^{(1)}\}_{t\geq 0}$,$\{N_t^{(2)}\}_{t\geq 0}$,$\{Y_i^{(1)}\}_{i\geq 1}$, $\{Y_i^{(2)}\}_{i\geq 1}$. In this case

$$\phi(z) = \frac{\sigma^2}{2}z^2 + \lambda_1\frac{z}{\alpha_1-z} - \lambda_2\frac{z}{\alpha_2+z}.$$

Proposition A.3.2. *The function $z \to \phi(z) - \epsilon$ has four non-zero roots ρ_i $(i = 1, 2, 3, 4)$ that satisfy $\rho_2 < -\alpha_2 < \rho_1 < 0 < \rho_3 < \alpha_1 < \rho_4$.*

From now on ρ_i, $(i = 1, 2, 3, 4)$ are these roots.

Proof. It is clear that the function $z \to \phi(z) - \epsilon$ has almost four real roots. Furthermore observe:

$$\lim_{z \to -\infty} \phi(z) - \epsilon = \infty, \quad \lim_{z \to -\alpha_2^-} \phi(z) - \epsilon = -\infty, \tag{37}$$

$$\lim_{z \to -\alpha_2^+} \phi(z) - \epsilon = \lim_{z \to \alpha_1^-} \phi(z) - \epsilon = \infty, \qquad \phi(0) - \epsilon = -\epsilon \tag{38}$$

$$\lim_{z \to \infty} \phi(z) - \epsilon = \infty, \quad \lim_{z \to \alpha_1^+} \phi(z) - \epsilon = -\infty. \tag{39}$$

Necessarily, from the term (37), we deduce there is a root $\rho_2 < -\alpha_2$. From the term (38), we obtain two roots $\rho_1 < 0 < \rho_3$ in the interval $(-\alpha_2, \alpha_1)$. Finally, from the term (39), we deduce there is root $\rho_4 > \alpha_1$. $\square$

Proposition A.3.3. *The matrix*

$$\mathbf{N}_{b-a} = \begin{pmatrix} 1 & 1 & e^{-\rho_1(a-b)} & e^{-\rho_2(a-b)} \\ \dfrac{1}{\alpha_1 - \rho_3} & \dfrac{1}{\alpha_1 - \rho_4} & \dfrac{e^{-\rho_1(a-b)}}{\alpha_1 - \rho_1} & \dfrac{e^{-\rho_2(a-b)}}{\alpha_1 - \rho_2} \\ e^{\rho_3(a-b)} & e^{\rho_4(a-b)} & 1 & 1 \\ \dfrac{e^{\rho_3(a-b)}}{\alpha_2 + \rho_3} & \dfrac{e^{\rho_4(a-b)}}{\alpha_2 + \rho_4} & \dfrac{1}{\alpha_2 + \rho_1} & \dfrac{1}{\alpha_2 + \rho_2} \end{pmatrix}$$

is always non-singular.

Proof. Let $x := e^{a-b} < 1$. The inequalities $\rho_2 < -\alpha_2 < \rho_1 < 0 < \rho_3 < \alpha_1 < \rho_4$ will be used without referencing them. We will prove that the transpose of $\mathbf{N}_{b-a}$ is invertible. That is, the matrix

$$\begin{pmatrix} 1 & \dfrac{1}{\alpha_1 - \rho_3} & x^{\rho_3} & \dfrac{x^{\rho_3}}{\alpha_2 + \rho_3} \\ 1 & \dfrac{1}{\alpha_1 - \rho_4} & x^{\rho_4} & \dfrac{x^{\rho_4}}{\alpha_2 + \rho_4} \\ x^{-\rho_1} & x^{\rho_4} & 1 & \dfrac{1}{\alpha_2 + \rho_3} \\ x^{-\rho_2} & \dfrac{x^{\rho_4}}{\alpha_2 + \rho_4} & 1 & \dfrac{1}{\alpha_2 + \rho_2} \end{pmatrix}.$$

By Gaussian elimination, it is equivalent to prove that the matrix

$$\begin{pmatrix} -\dfrac{1}{\alpha_1 - \rho_3} + \dfrac{1}{\alpha_1 - \rho_4} & x^{\rho_4} - x^{\rho_3} & \dfrac{x^{\rho_4}}{\alpha_2 + \rho_4} - \dfrac{x^{\rho_3}}{\alpha_2 + \rho_3} \\ x^{-\rho_1}\left(\dfrac{1}{\alpha_1 - \rho_1} - \dfrac{1}{\alpha_1 - \rho_3}\right) & 1 - x^{-\rho_1 + \rho_3} & \dfrac{1}{\alpha_2 + \rho_1} - \dfrac{x^{\rho_3 - \rho_1}}{\alpha_2 + \rho_3} \\ x^{-\rho_2}\left(\dfrac{1}{\alpha_2 - \rho_2} - \dfrac{1}{\alpha_1 - \rho_3}\right) & 1 - x^{\rho_3 - \rho_2} & \dfrac{1}{\alpha_2 + \rho_2} - \dfrac{x^{\rho_3 - \rho_2}}{\alpha_2 + \rho_3} \end{pmatrix}, \tag{40}$$

is invertible. Using Cramer's rule, we express its determinant as:

$$\left(-\frac{1}{\alpha_1-\rho_3}+\frac{1}{\alpha_1-\rho_4}\right)\left(1-x^{\rho_3-\rho_1}\right)\left(\frac{1}{\alpha_2+\rho_2}-\frac{x^{\rho_3-\rho_2}}{\alpha_2+\rho_3}\right) \tag{41}$$

$$+\left(x^{\rho_4}-x^{\rho_3}\right)\left(\frac{1}{\alpha_2+\rho_1}-\frac{x^{\rho_3-\rho_1}}{\alpha_2+\rho_3}\right)\left(\frac{1}{\alpha_1-\rho_1}-\frac{1}{\alpha_1-\rho_3}\right)x^{-\rho_2} \tag{42}$$

$$+\left(\frac{x^{\rho_4}}{\alpha_2+\rho_4}-\frac{x^{\rho_3}}{\alpha_2+\rho_3}\right)\left(\frac{1}{\alpha_1-\rho_1}-\frac{1}{\alpha_1-\rho_3}\right)\left(1-x^{\rho_3-\rho_2}\right)x^{-\rho_1} \tag{43}$$

$$-\left(\frac{1}{\alpha_1-\rho_2}-\frac{1}{\alpha_1-\rho_3}\right)\left(1-x^{\rho_3-\rho_1}\right)\left(\frac{x^{\rho_4}}{\alpha_2+\rho_4}-\frac{x^{\rho_3}}{\alpha_2+\rho_3}\right)x^{-\rho_2} \tag{44}$$

$$-\left(1-x^{\rho_3-\rho_2}\right)\left(\frac{1}{\alpha_2+\rho_1}-\frac{x^{\rho_3-\rho_1}}{\alpha_2+\rho_3}\right)\left(-\frac{1}{\alpha_1-\rho_3}+\frac{1}{\alpha_1-\rho_4}\right) \tag{45}$$

$$-\left(\frac{1}{\alpha_2+\rho_2}-\frac{x^{\rho_3-\rho_2}}{\alpha_2+\rho_3}\right)\left(\frac{1}{\alpha_1-\rho_1}-\frac{1}{\alpha_1-\rho_3}\right)(x^{\rho_4}-x^{\rho_3})x^{-\rho_1}. \tag{46}$$

We proceed to prove that the expression is always positive if $0 < x < 1,\ \rho_2 < -\alpha_2 < \rho_1 < 0 < \rho_3 < \alpha_1 < \rho_4$. First, lets examine the term (41) plus the term (44):

$$\left(1-x^{\rho_3-\rho_1}\right)\Bigg(\left(-\frac{1}{\alpha_1-\rho_3}+\frac{1}{\alpha_1-\rho_4}\right)\left(\frac{1}{\alpha_2+\rho_2}-\frac{x^{\rho_3-\rho_2}}{\alpha_2+\rho_3}\right) - \left(\frac{1}{\alpha_1-\rho_2}-\frac{1}{\alpha_1-\rho_3}\right)\left(\frac{x^{\rho_4}}{\alpha_2+\rho_4}-\frac{x^{\rho_3}}{\alpha_2+\rho_3}\right)x^{-\rho_2}\Bigg) \tag{47}$$

First observe, $(1-x^{\rho_3-\rho_1}) > 0,\ 0 < x^{-\rho_2} < 1$. Thus, to prove that (47) is positive, it is enough to show that the expression

$$\left(-\frac{1}{\alpha_1-\rho_3}+\frac{1}{\alpha_1-\rho_4}\right)\left(\frac{1}{\alpha_2+\rho_2}-\frac{x^{\rho_3-\rho_2}}{\alpha_2+\rho_3}\right) - \left(\frac{1}{\alpha_1-\rho_2}-\frac{1}{\alpha_1-\rho_3}\right)\left(\frac{x^{\rho_4}}{\alpha_2+\rho_4}-\frac{x^{\rho_3}}{\alpha_2+\rho_3}\right) \tag{48}$$

is positive. For that endeavor, observe that all the four terms inside the parentheses are negative, thus it is enough that

$$\begin{gathered}\frac{-1}{\alpha_1-\rho_4}+\frac{1}{\alpha_1-\rho_3} > \frac{1}{\alpha_1-\rho_3}-\frac{1}{\alpha_1-\rho_2},\\ \frac{x^{\rho_3-\rho_2}}{\alpha_2+\rho_3}-\frac{1}{\alpha_2+\rho_2} > \frac{x^{\rho_4-\rho_2}}{\alpha_2+\rho_3}-\frac{x^{\rho_4-\rho_2}}{\alpha_2+\rho_4}.\end{gathered}$$

Which are true because $\alpha_1-\rho_4 < 0,\ \alpha_1-\rho_2 > 0$ and $\alpha_2+\rho_2 > 0,\ \alpha_2+\rho_4 > 0$. Therefore the term (41) plus the term (44) is positive. Lets prove that the (42) plus the term (46) is positive too, that is the expression:

$$(x^{\rho_4}-x^{\rho_3})\left(\frac{1}{\alpha_1-\rho_1}-\frac{1}{\alpha_1-\rho_3}\right)\times \\ \left(x^{-\rho_2}\left(\frac{1}{\alpha_2+\rho_1}-\frac{x^{\rho_3-\rho_1}}{\alpha_2+\rho_3}\right)-x^{-\rho_1}\left(\frac{1}{\alpha_2+\rho_2}-\frac{x^{\rho_3-\rho_2}}{\alpha_2+\rho_3}\right)\right). \quad (49)$$

The product in the first line is positive because both terms are negative. What is left to do is that the second line is positive. For that endeavor, observe both $\alpha_2+\rho_1$ and $-(\alpha_2+\rho_2)$ are positive and the rest of the terms cancel each other. Therefore (42) plus the term (46) is positive. We proceed to show (43) is positive. This case is simple because $(1-x^{\rho_3-\rho_2})x^{-\rho_1}>0$ and the other two terms are negative because $x^{\rho_4}<x^{\rho_3}$, $\alpha_2+\rho_4>\alpha_2+\rho_3$ and $\alpha_1-\rho_1>0$, $\alpha_1-\rho_3>0$. Finally the term (45) is positive because $(1-x^{\rho_3-\rho_2})>0$, $x^{\rho_3-\rho_1}<1$, $\alpha_2+\rho_1<\alpha_2+\rho_3$ and $\alpha_1-\rho_4<0$, $-(\alpha_1-\rho_3)<0$. We conclude that the determinant is positive and deduce that the matrix is always non-singular. □

Unlike the case without Gaussian Noise, it is not obvious how to construct an adequate martingale. We need a couple of propositions before using the matrix defined previously.

Proposition A.3.4. *For every $z\in i\mathbb{R}$, $a>0$ the stochastic process:*

$$M_t=e^{-at+zX_t}-1-(\log(\phi(z))-a)\int_0^t e^{-as+zX_s}ds,$$

is a martingale.

Proof. Take $0\le u\le t$ and observe:

$$\begin{aligned}\mathbf{E}(M_t|\mathcal{F}_u)&=e^{-au+zX_u}\mathbf{E}(e^{-a(t-u)+z(X_t-X_u)})-1\\&-(\log(\phi(z))-a)\int_0^u e^{-as+zX_s}ds-(\log(\phi(z))-a)\int_u^t e^{-au+zX_u}\mathbf{E}(e^{-a(s-u)+z(X_{s-u})})ds\\&=e^{-au+zX_u}\mathbf{E}(e^{-a(t-u)+z(X_{t-u})})-1\\&-(\log(\phi(z))-a)\int_0^u e^{-as+zX_s}ds-(\log(\phi(z))-a)e^{-au+zX_u}\int_0^{t-u}\mathbf{E}(e^{-a+zX_1})^r dr\\&=M_u.\end{aligned}$$

The last equality comes from the fact $\int k^r dr=k^r/\log(k)$ for k complex with positive real part. □

Theorem A.3.5. *Let $a < 0 < b$, γ the first exit of the interval (a, b). Consider any non-negative measurable function f such that $\int_0^\infty f(y+b)e^{-\alpha_1 y}dy$ and $\int_{-\infty}^0 f(y+a)e^{\alpha_2 y}dy$ are integrable. For any $\epsilon > 0$, we have:*

$$\mathbf{E}f(X_\gamma)e^{-\epsilon\gamma} = (e^{-\rho_3 b}, e^{-\rho_4 b}, e^{-\rho_1 a}, e^{-\rho_2 a})\mathbf{N}_{b-a}^{-1}\mathbf{f},$$

where $\mathbf{f} = (f_0^u, f_1^u, f_0^d, f_1^d)^t$ with

$$f_0^u = f(b), \qquad f_1^u = \int_0^\infty f(y+b)e^{-\alpha_1 y}dy,$$
$$f_0^d = f(a), \qquad f_1^d = \int_{-\infty}^0 f(y+a)e^{\alpha_2 y}.$$

Proof. When the process leaves the interval it might cross the boundaries, due to the gaussian part, or jump above/below them, due to the jumps of the process. We define these events:

$$F_0 = \{X_\gamma = b\}, \qquad G_0 = \{X_\gamma = a\},$$
$$F_1 = \{X_\gamma > b\}, \qquad G_1 = \{X_\gamma < a\}.$$

Using the loss of memory property of the exponential distribution:

$$\mathbf{E}(e^{-\epsilon\gamma}f(X_\gamma)) = \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{F_0}f(b) + \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{F_1}f_1^u\alpha_1 + \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{G_0}f(a) + \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{G_1}f_1^d\alpha_2. \tag{50}$$

On the other hand from Proposition A.3.4, we have for every $z \in i\mathbb{R}$:

$$\begin{aligned}0 = \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{F_0}e^{zb} + \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{F_1}e^{zb}\frac{\alpha_1}{\alpha_1 - z} \\ + \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{G_0}e^{za} + \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{G_1}e^{za}\frac{\alpha_2}{\alpha_2 + z} \\ - 1 - (\log(\phi(z)) - \epsilon)\mathbf{E}\int_0^\gamma \exp(-\epsilon s + zX_s)ds.\end{aligned} \tag{51}$$

The right hand side of the equation defines a function H for $z \in i\mathbb{R}$. Observe that the function defined as $z \to (\alpha_1 - z)(\alpha_2 + z)H(z)$ can be extended to all the complex numbers. Moreover its value is zero in all the imaginary numbers, thus the function is always zero. Therefore H is zero in every $z \in \mathbb{C} - \{\alpha_1, \alpha_2\}$. By using the fact ρ_i, $i = 1, 2, 3, 4$ are the roots of $\log(\phi(z)) - \epsilon$ we get that when $z = \rho_i$ (51) can be rewritten as:

$$1 = \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{F_0}e^{\rho_i b} + \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{F_1}e^{\rho_i b}\frac{\alpha_1}{\alpha_1 - \rho_i} + \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{G_0}e^{\rho_i a} + \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{G_1}e^{\rho_i a}\frac{\alpha_2}{\alpha_2 + \rho_i}.$$

Which implies that the next matrix equation holds:

$$\left(\mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{F_0} \quad \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{F_1}\alpha_1 \quad \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{G_0} \quad \mathbf{E}e^{-\epsilon\gamma}\mathbf{1}_{G_1}\alpha_2\right)\mathbf{N}_{b-a} = (e^{-\rho_3 b}, e^{-\rho_4 b}, e^{-\rho_1 a}, e^{-\rho_2 a}).$$

Multiplying the equality at the right by $\mathbf{N}_{b-a}^{-1}$ $\mathbf{f}$ and using equation (50) we conclude the proof. $\square$

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